Logistic Map

$$x_{n+1} = r x_n (1 - x_n)$$

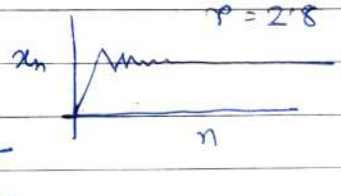
This is a discrete-time analog of the logistic eqn of Population growth.

Here, x_n is the dimensionless measure of the population in the n^{th} generation, and $r \geq 0$ is the intrinsic growth rate.

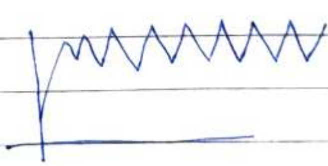
Value of r is usually restricted to $0 \leq r \leq 4$ so that the map, maps the interval $0 \leq x \leq 1$. For $r > 4$, the map may show negative population, which is absurd.

For small growth rate $r < 1$, the population always goes extinct: $x_n \to 0$ as $n \to \infty$.

For $1 < r < 3$, population grows and eventually reaches a nonzero steady state.

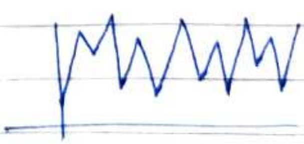


For $r > 3$, the population builds up again but now oscillates about the former steady state. Oscillating betn a large population in one generation and a smaller population in the next.



This type of oscillation in which x_n repeats every two iterations, is called a period-2 cycle.

At $r > 3.5$, the population repeats itself after 4 generations. The previous cycle is doubled to period-4.



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Further period-doubling occurs as r increases. Specifically where a 2^n cycle is born.

$r_1 = 3$ period-2

$r_2 = 3.449$ 4

$r_3 = 3.54409$ 8

$r_4 = 3.5644$ 16

$\vdots$

$r_\infty = 3.569946$ ∞

Actually, $\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669.$

That is the ratio of the distances betn successive transitions is a constant.

Predator-prey dynamics :-

2D - Lotka-Volterra System :-

Assumptions :-

1) The prey population increases exponentially in the absence of predators. This assumption is highly simplified. A better approximation could be the logistic growth. However we will stick to the exponential growth.

2) The predator population decreases exponentially in the absence of prey. Ofcourse we are neglecting other potential reasons for the decay of predator like environmental factors. ~~dieses~~ dieses.

3) The prey population decreases relative to the frequency with which predators meet prey as a result of predation.

4) The predator population increases relative to the frequency with which predators meet prey as a result of predation.

Based on these assumption the Lotka-Volterra equations for predator-prey dynamics can be given by,

$$\dot{x} = \alpha x - \beta xy$$
$$\dot{y} = -\gamma y + \delta xy$$

$x \rightarrow$ prey population
$y \rightarrow$ predator population

$\alpha, \beta, \gamma, \delta$ are +ve parameters

The αx term models consumption #1
$-\gamma y$ " " " #2
$-\beta xy$ " " " #3
δxy " " " #4

$$\therefore \quad \dot{x} = \alpha x - \beta xy$$
$$\dot{y} = -\gamma y + \delta xy$$

The fixed points of the system are,

$$\dot{x} = \alpha x - \beta xy = 0$$
$$\Rightarrow x(\alpha - \beta y) = 0$$
$$\Rightarrow x = 0 \quad \text{and} \quad y = \alpha/\beta$$

$$\dot{y} = -\gamma y + \delta xy = 0$$
$$\Rightarrow y(-\gamma + \delta x) = 0$$
$$\Rightarrow y = 0 \quad \text{and} \quad x = \gamma/\delta$$

$\therefore$ Fixed points are, $(x^*, y^*) = \begin{cases} (0,0) \\ \cancel{(\alpha/\beta, \gamma/\delta)} \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) \end{cases}$

Linear stability analysis.

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{\mathbf{a}}$$

$$= \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & -\gamma + \delta x \end{pmatrix}$$

$$J\big|_{(0,0)} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}, \quad \Delta = -\alpha\gamma < 0$$

$\therefore \quad \lambda_1 = \alpha, \quad \lambda_2 = -\gamma$

~~Since, $\alpha\, \delta\, \beta > 0$, therefor (0,0) fixed.~~

Since, both $\alpha > 0$ & $\gamma > 0$, these eigenvalues indicate $(0,0)$ is a saddle point.

$$J\big|_{\left(\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right)} = \begin{pmatrix} \alpha - \frac{\beta\gamma}{\delta} & -\frac{\beta\alpha}{\beta} \\ \delta\frac{\gamma}{\delta} & -\gamma + \delta\frac{\alpha}{\beta} \end{pmatrix}$$

$$J\big|_{\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)} = \begin{pmatrix} \alpha - \beta\frac{\alpha}{\beta} & -\beta\frac{\gamma}{\delta} \\ \delta\frac{\alpha}{\beta} & -\gamma + \delta\frac{\gamma}{\delta} \end{pmatrix}$$

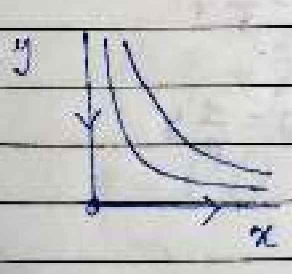
$$= \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \alpha\delta/\beta & 0 \end{pmatrix}$$

$\lambda^2 = \frac{\beta\gamma}{\delta} \cdot \frac{\alpha\delta}{\beta}$
$= \alpha\gamma$
$\lambda = \pm i\sqrt{\alpha\gamma}$
$\therefore \omega = \sqrt{\alpha\gamma}$

$\tau = 0, \quad \Delta = \frac{\beta\gamma}{\delta} \cdot \frac{\alpha\delta}{\beta} = \alpha\gamma > 0$

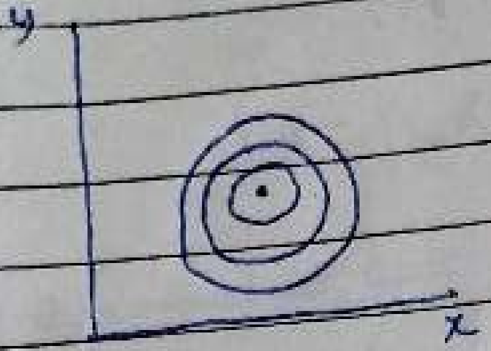
$\therefore \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ is a centre.

$\lambda_1 = \alpha, \; (-\gamma - \alpha)x_2 = 0$
$\Rightarrow x_2 = 0$
$\therefore \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Growing direction

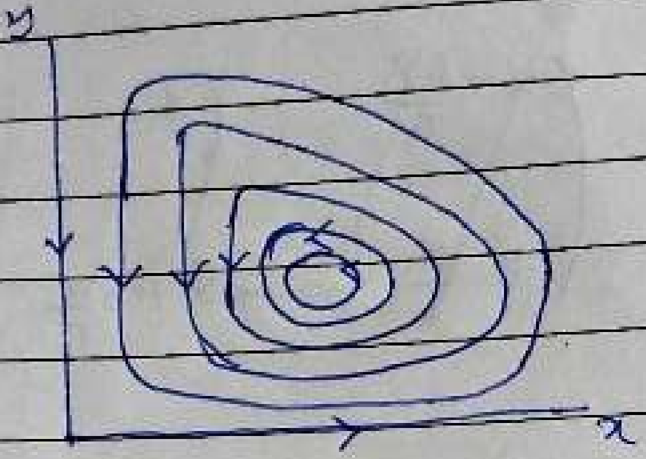


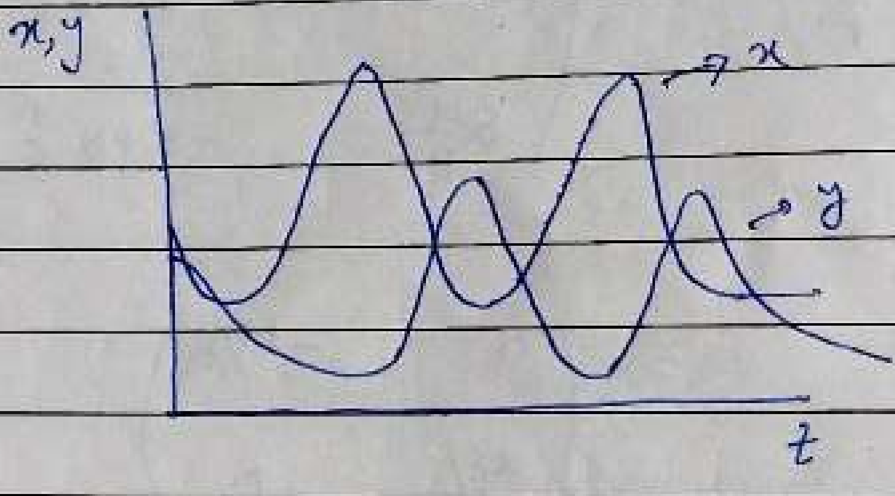
$\lambda_2 = -\gamma, \; (\alpha + \gamma)x_1 = 0$
$\Rightarrow x_1 = 0$
$\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ decaying direction

And about $(\gamma/\delta, \alpha/\beta)$



Together,





Nondimensionalization.

$$t' = \sqrt{\alpha\gamma}\, t \quad \Rightarrow \quad \frac{d}{dt} = \sqrt{\alpha\gamma}\,\frac{d}{dt'}$$

$$x' = \frac{x}{x^*} = \frac{\delta}{\gamma}x \quad \Rightarrow x = \frac{\gamma}{\delta}x'$$

$$y' = \frac{y}{y^*} = \frac{\beta y}{\alpha} \quad \Rightarrow y = \frac{\alpha}{\beta}y'$$

$\therefore$ $\dot{x} = \alpha x - \beta xy$

$$\frac{\gamma}{\delta}\sqrt{\alpha\gamma}\,\dot{x}' = \alpha \cdot \frac{\gamma}{\delta}x' - \beta\frac{\gamma}{\delta}\frac{\alpha}{\beta}x'y'$$

$$\dot{x}' = \sqrt{\frac{\alpha}{\gamma}}\,x' - \sqrt{\frac{\alpha}{\gamma}}\,x'y'$$

$$\dot{x}' = \sqrt{\frac{\alpha}{\gamma}}\,(x' - x'y') = r\,(x' - x'y')$$

$$r = \sqrt{\alpha/\gamma}$$

$\dot{y} = -\gamma y + \delta xy$

$$\frac{\alpha}{\beta}\sqrt{\alpha\gamma}\,\dot{y}' = -\gamma\frac{\alpha}{\beta}y' + \delta\frac{\gamma}{\delta}\frac{\alpha}{\beta}x'y'$$

$$\dot{y}' = -\sqrt{\frac{\gamma}{\alpha}}\,y' + \sqrt{\frac{\gamma}{\alpha}}\,x'y'$$

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$$\therefore \dot{y}' = \frac{1}{r}(-y' + x'y')$$

$\therefore$ Non dimensional eqn.

$$\dot{x}' = r(x' - x'y')$$
$$\dot{y}' = \frac{1}{r}(-y' + x'y')$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\gamma y + \delta xy}{\alpha x - \beta xy} = \frac{y(-\gamma + \delta x)}{x(\alpha - \beta y)}$$

$$\frac{\alpha - \beta y}{y} dy = \frac{-\gamma + \delta x}{x} dx$$

Integrating, $\alpha \ln y - \beta y = -\gamma \ln x + \delta x + C$

or, $\alpha \ln y + \gamma \ln x = \delta x + \beta y + C$

where C is a constant. For each value of C, the trajectory is an implicit curve which is an isolevel or level set of the function

$$F(x,y) = \alpha \ln y + \gamma \ln x - \delta x - \beta y = \text{constant}$$

$$F\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \alpha \ln \frac{\alpha}{\beta} + \gamma \ln \frac{\gamma}{\delta} - \delta \frac{\gamma}{\delta} - \beta \frac{\alpha}{\beta}$$
$$= \alpha \ln \frac{\alpha}{\beta} + \gamma \ln \frac{\gamma}{\delta} - \delta - \beta$$

If we had a spiral node or repeller, then the critical point would lie on many of these level sets. But then $F\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ would have many values, which makes no sense, since F is continuous at $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$. The non linear system therefore has stable centre.

Quantifying Chaos

Lyapunov Exponents

Suppose we have an one dimensional system

$$\dot{x}(t) = f(x)$$

Let x_0 be an initial point & x be another initial point nearby. Let $x_0(t)$ be the trajectory arising from x_0 & $x(t)$ be the trajectory arising from x.

Therefore the rate of change of the distance betn two trajectories is given by

$$\dot{s} = \dot{x} - \dot{x}_0$$
$$= f(x) - f(x_0)$$

If, x is close to x_0, then we can use Taylor expansion to write,

$$\dot{s} = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x-x_0) + \cdots - f(x_0)$$

$$\approx \left.\frac{df}{dx}\right|_{x=x_0} (x-x_0)$$

$$\approx \left(\frac{df}{dx}\right)_{x_0} s$$

We defined Lyapunov exponent as ~~the rate of change of the distance betn trajectories~~

$$\cancel{s(t) = s e^{\lambda t}}$$

$$s(t) = s(0)\, e^{\lambda t}$$

$$\therefore \dot{s} = \lambda\, s(t=0)\, e^{\lambda t} = \lambda s$$

$$\therefore \lambda = \left.\frac{df}{dx}\right|_{x_0}$$

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Thus we see that if λ is +ve, then the two trajectories diverge and if λ is negative the trajectories converge.

In 2D or more dimension we can associate a Lyapunov exponent with the rate of expansion or contraction of trajectories for each dimension in the state space. In 3D we define Lyapunov exponents which are the e.values of the Jacobian matrix evaluated at the phase space point in question.

In special case where the Jacobian matrix is zero everywhere except the diagonal the three e.values and hence the three local Lyapunov exponents are given by

$$\lambda_1 = \frac{\partial f_1}{\partial x}, \quad \lambda_2 = \frac{\partial f_2}{\partial y}, \quad \lambda_3 = \frac{\partial f_3}{\partial z}$$

We can calculate the Lyapunov exponent over the whole time history of the trajectory. This would allow us to compute the average Lyapunov exponents for the system. If at least one of the average Lyapunov exponent is +ve, then the attractor is chaotic.

Sign of λ's	Type of attractor
$(-,-,-)$	F.P.
$(0,-,-)$	Limit Cycle
$(0,0,-)$	Quasi Periodic
$(+,0,-)$	Chaotic

From Time Series

Suppose we have an one dimensional time series of data.

$$x(t_0), x(t_1), x(t_2) \cdots \equiv x_0, x_1, x_2, \cdots$$

We will also assume that the time intervals betn samples are all equal, i.e.

$$t_n - t_0 = n\tau$$

where τ is the interval betn samples.

If the system is behaving chaotically, the divergence of nearby trajectories will manifest itself in the following way: If we select some value from the sequence of x's say x_i, then search the sequence for another x value, say x_j, that is close to x_i, then the sequence of differences,

$$d_0 = |x_j - x_i|$$
$$d_1 = |x_{j+1} - x_{i+1}|$$
$$d_2 = |x_{j+2} - x_{i+2}|$$
$$\vdots$$
$$d_n = |x_{j+n} - x_{i+n}|$$

is assumed to increase exponentially, at last on the average as n increases.

i.e. we assume, $d_n = d_0 e^{\lambda n}$

i.e. $\lambda = \frac{1}{n} \ln \frac{d_n}{d_0}$

If, λ is +ve then the trajectory is chaotic.

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Caveats

1) We assumed that the nearby trajectories separate exponentially. To test that we can plot the logarithm of d_m with m. If the divergence is exponential the points will fall on a straight line, the slope of which is Lyapunov exponent. If the data do not fall close to the straight line then the quoted L.E. is meaningless.

2). The value of λ may depend on the value of x_i chosen as the initial value, i.e. λ is func. of x_i $\lambda(x_i)$. To characterize λ we want the average value for λ. We do that by finding $\lambda(x_i)$ for large no. of x_i's and then averaging it.

$$\lambda = \frac{1}{N}\sum_{i=1}^{N} \lambda(x_i)$$

```
N = 10000
ij = 500
L = np.zeros(int(len(x)/N) - 1)
K = np.arange(0, len(x) - N, N)
for i, n in enumerate(K):
    d0 = np.fabs(x[n+ij] - x[n])
    dN = np.fabs(x[n+ij+N] - x[n+N])
    L[i] = np.log(dN/d0)/N

print("Average Lyapunov exponent =", np.average(L))
```

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Routes to Chaos

1. Period Doubling
2. Quasi-Periodicity
3. Intermittency
4. Chaotic transients
5. Crises

Period Doubling Route to Chaos :-

The period doubling route begins with limit cycle behaviour. This limit cycle may have been born through a Hopf bifurcation from a node or other fixed point. As some control parameter changes this limit cycle becomes unstable. If the limit cycle becomes unstable by having one of its characteristic multipliers become more -ve than -1 (i.e. $|M| > 1$), then the new motion remains periodic, but has a period twice as long as the period of the original motion. In the Poincare section, this new limit cycle exhibits two points one on each side of the original Poincare section point.

This alteration of ~~limit cycle~~ intersection points is related to the characteristic multiplier associated with the original limit cycle, which has gotten more negative than -1. Since $|M| > 1$, the trajectory's map points are now being repelled by the original map point. The minus sign tells us that they alternate from one side to the other, as we can see from the defn. of Floquet multiplier.

$$d_2 = M d_1$$

Hence this kind of bifurcation is also called flip bifurcation.

As the control parameter is changed further, this period-2 limit cycle may become unstable and give birth to a period-4 cycle with four Poincare section points.

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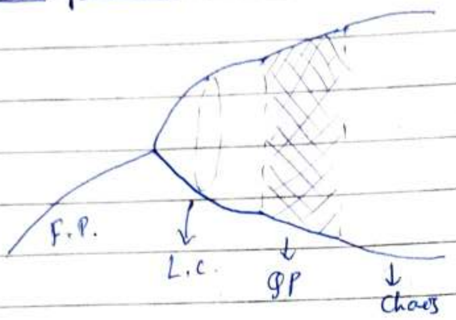
The period doubling process may continue until the period becomes infinite, that is the trajectory never repeats itself. The trajectory is then chaotic.

Quasiperiodic Route to Chaos



The system may start with a fixed point attractor. As a control parameter is changed the system may undergo a Hopf bifurcation to ~~by~~ develop periodic behaviour characterised by limit cycle in state space. A second frequency may appear with a further change in control parameter. The state space trajectories then reside on the surface of a torus. If the second frequency is incommensurate with the first, then the trajectory eventually covers the surface of the torus. Further changes in the control parameter then lead to the introduction of a third frequency. The phase space trajectories then live on a 3D torus. With further changes the system's behaviour may become chaotic.

Intermittency

Intermittency occurs whenever the behaviour of a system seems to switch back and forth betn two qualitatively different behaviours even though all the control parameters remain constant. This switching seems to occur randomly. There are two types. In the first type the system's behaviour seem to switch betn periodic behaviour and chaotic behaviour. The behaviour of the system is predominantly periodic for some parameter value with occasional bursts of chaotic behaviour. As the control parameter value is changed, the time spent being chaotic increases and the time spent being chaotic decreases until the behaviour is chaotic all the time.

In the second type the system's behaviour seems to switch betn periodic and quasiperiodic behaviour.

In type-I intermittency the Floquet multiplier crosses the unit circle along the real axis at +1. This leads to irregularly occuring bursts of periodic & chaotic behaviour.

If the two Floquet multipliers form a complex conjugate pair, then the imaginary part indicates the presence of a second frequency in the behaviour of the system. (Hopf bifurcation of ~~Hopf bifur~~ Limit Cycle). At the bifurcation point the limit cycle corresponding to the second frequency becomes unstable, and we observe bursts of two frequency behaviour mixed with intervals of chaotic behaviour. This is type-II intermittency.

If the Floquet multiplier is -ve and become more -ve than -1, then a type of period doubling bifurcation takes place. However this periodic behaviour is interrupted with bursts of chaotic behaviour. This is type-3 intermittency or period doubling intermittency.