Laplace Transform

Class 1

Integral Transform:

Frequently in mathematical physics, we encounter pairs of functions related by an expression of the form:

$$g(x) = \int_a^b f(t)\, K(x,t)\, dt. \quad -(1)$$

where it is understood 'a', 'b' and 'K(x,t)' will be the same for all function pairs f and g.

We can write the relationship in a more symbolic form,

$$g(x) = \mathcal{L} f(t)$$

thereby interpreting equation (1) as an operator equation.

The function $g(x)$ is called the integral transform of $f(t)$ by the operator $\mathcal{L}$, with the specific transform determined by the choice of 'a', 'b' and $K(x,t)$.

In order for transforms to be useful from a practical viewpoint, there must exist an operator $\mathcal{L}^{-1}$ and also a reasonably convenient and powerful method of evaluating

$$\mathcal{L}^{-1} g(x) = f(t)$$

for an acceptably broad range of $g(x)$.

** Why are Integral Transforms usefull ??

① The most common applications are in situations where we have a problem that can be solved only with difficulty, if at all, in its original formulation. Howere, it may happen that the transform of the problem can be solved relatively easily.

$$\begin{tikzcd}
\text{Problem in transform space (2)} \arrow[r, "\text{Relatively easy solution}"] & \text{Solution in transform space (3)} \arrow[d, "\text{Inverse Transform}"] \\
\text{Original problem ①} \arrow[u, "\text{Integral Transform}"] \arrow[r, dashed, "\text{DIFFICULT}", "\text{SOLUTION}"'] & \text{Solution of original problem (4)}
\end{tikzcd}$$

(2) Anoter frequent use is, ~~to~~ together with its inverse, to form an integral representation of a function that we originally had in an explicit form. This ~~happe~~ proves helpfull due to the relatively simple behaviour of the transform of differentiation and integration operator.

The Integral Transfom of widest use is Fourier Transforme defined by,

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt.$$

The second transform, which we will be reading in detail is the Laplace Transform given by,

$$\boxed{F(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt}$$

* Laplace Transform of elementary functions:

(*) $$\boxed{L[f(t)] = F(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt}$$

(1) $L(1) = ??$

$$L(1) = \int_{0}^{\infty} 1 . e^{-st}\, dt$$

$$= \left.\frac{e^{-st}}{-s}\right|_{0}^{\infty}$$

$$= \frac{1}{s} - 0$$

$$= \frac{1}{s}$$

$$\boxed{L(1) = \frac{1}{s}}$$

② $L(e^{at}) = ?$

Soln:

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \frac{1}{s-a}$$

$$\boxed{L(e^{at}) = \frac{1}{s-a}}$$

③ $L(t^n) = ?$

$$L(t^n) = \int_0^\infty e^{-st} t^n dt$$

Putting, $st = x$

$s dt = dx$

$$\therefore L(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx$$

$$\left[\because \int_0^\infty e^{-x} x^n dx = \Gamma(n+1)\right]$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

$$= \frac{n!}{s^{n+1}}$$

$$\boxed{L(t^n) = \frac{n!}{s^{n+1}}}$$

④ $L(\cos at) = ??$

Soln:

We know,

$$\cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$\left[\begin{aligned} \because e^{iat} &= \cos at + i\sin at \\ e^{-iat} &= \cos at - i\sin at \end{aligned}\right]$$

$$L(\cos at) = L\left(\frac{e^{iat}}{2} + \frac{e^{-iat}}{2}\right)$$

$$= \frac{1}{2}L(e^{iat}) + \frac{1}{2}L(e^{-iat})$$

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2+a^2}\right]$$

$$= \frac{s}{s^2+a^2}$$

$$\boxed{L(\cos at) = \frac{s}{s^2+a^2}}$$

similarly, it can be ~~calculated~~ obtained,

(5) $\boxed{L(\sin at) = \frac{a}{s^2+a^2}}$

(6) $L(\cosh at) = ??$

solns:

We know,

$$\cosh at = \frac{e^{at}+e^{-at}}{2}$$

$$\therefore L(\cosh at) = \frac{1}{2}\left[L(e^{at}) + L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{s}{s^2-a^2}$$

$$\boxed{L(\cosh at) = \frac{s}{s^2-a^2}}$$

similarly, it can be found out,

$$\boxed{L(\sinh at) = \frac{a}{s^2-a^2}}$$

Properties of L.T:

1. $L[af_1(t) + bf_2(t)] = aL[f_1(t)] + bL[f_2(t)]$

Proof:

$$L[af_1(t) + bf_2(t)] = \int_0^\infty e^{-st}[af_1(t) + bf_2(t)]\,dt$$

$$= a\int_0^\infty e^{-st} f_1(t)\,dt + b\int_0^\infty e^{-st} f_2(t)\,dt$$

$$= aL[f_1(t)] + bL[f_2(t)] \quad \text{(proved)}$$

2. First Shifting Theorem:

If $L[f(t)] = F(s)$

then, $L[e^{at} f(t)] = F(s-a)$

Proof: $$L[e^{at} f(t)] = \int_0^\infty e^{-st} \cdot e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

Let $s-a = r$

$$= \int_0^\infty e^{-rt} f(t)\,dt$$

$$= F(r)$$

$$= F(s-a) \quad \text{(proved)}$$

H.W. Find the L.T of

1. $\{e^{at} \sin bt\}$
2. $e^{-t}\cos^2 t$

Class II

3. **Laplace Transform of the derivative of $f(t)$:**

$$L[f'(t)] = sL[f(t)] - f(0)$$

Proof:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$$

Integrating by parts,

$$L[f'(t)] = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\, dt$$

$$= 0 - 1.f(0) + s\int_0^\infty e^{-st} f(t)\, dt$$

$$\boxed{L[f'(t)] = sL[f(t)] - f(0)}$$

4. **LT of derivative of order n:** $\boxed{L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{n-1}(0)}$

$$L[f^n(t)] = sL[f^{n-1}(t)] - f^{n-1}(0)$$

Method 1:

$$= s\{sL[f^{n-2}(t)] - f^{n-2}(0)\} - f^{n-1}(0)$$

$$= s^2 L[f^{n-2}(t)] - sf^{n-2}(0) - f^{n-1}(0)$$

$$= s^2\{sL[f^{n-3}(t)] - f^{n-3}(0)\} - sf^{n-2}(0) - f^{n-1}(0)$$

$$= s^3 L[f^{n-3}(t)] - s^2 f^{n-3}(0) - sf^{n-2}(0) - f^{n-1}(0)$$

$\vdots$

$$= s^j L[f^{n-j}(t)] - s^{j-1} f^{n-j}(0) - s^{j-2} f^{n-(j-1)}(0) \ldots f^{n-1}(0).$$

$\vdots$

$j = n$

$$= s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) + \ldots f^{n-1}(0).$$

$$\boxed{L[f^n(t)] = s^n L[f(t)] - [s^{n-1} f(0) + s^{n-2} f'(0) + \ldots f^{n-1}(0)]}$$

Method 2:

We know,

$$L[f'(t)] = sL[f(t)] - f(0)$$

Again,

$$L[f''(t)] = sL[f'(t)] - f'(0)$$
$$= s\left[sL\{f(t)\} - f(0)\right] - f'(0)$$
$$= s^2 L[f(t)] - sf(0) - f'(0)$$

Similarly,

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - s^3 f'(0) - f''(0)$$

Similarly, ⋮

$$\boxed{L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \dots - f^{n-1}(0)}$$

2. <u>LT of integral of f(t):</u>

$$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} \qquad \text{where } L[f(t)] = F(s)$$

Proof: Let $\phi(t) = \int_0^t f(t)\,dt$ and $\phi(0) = 0$.

then, $\phi'(t) = f(t)$

We know,

$$L[\phi'(t)] = sL[\phi(t)] - \phi(0)$$

or, $L[f(t)] = sL\left[\int_0^t f(t)\,dt\right] - 0$

or, $\boxed{L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}}$ where $L[f(t)] = F(s)$

example:

(a) Find the L.T of $\int_0^t t^n e^{-at}\,dt$

Soln: $L[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}}$

$$\therefore L\left[\int_0^t t^n e^{-at}\,dt\right] = \frac{1}{s} \cdot \frac{n!}{(s+a)^{n+1}} \text{ (Ans)}$$

6. **Derivative of Laplace Transform:**

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$ where $F(s) = L[f(t)]$

Proof:

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\int_0^\infty f(t)\, e^{-st}\, dt\right]$$

$$= \int_0^\infty \left\{\frac{\partial}{\partial s}(e^{-st})\right\} f(t)\, dt$$

$$= \int_0^\infty (-te^{-st}) . f(t)\, dt$$

$$\frac{d}{ds}[F(s)] = L[-t f(t)] \quad \text{or,} \quad L[t f(t)] = (-1)^1 \frac{d}{ds}[F(s)]$$

Similarly,

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[F(s)]$$

Similarly,

$$\boxed{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]}$$

Question: Find the L.T of $t \cos at$.

Soln:

$L[t \cos at] = ??$

We know, $L[\cos at] = \frac{s}{s^2+a^2}$

Now, we have,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

$$\therefore L[t \cos at] = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= -\left[\frac{-s.2s}{(s^2+a^2)^2} + \frac{1}{s^2+a^2}\right]$$

$$= -\left[\frac{-2s^2+s^2+a^2}{(s^2+a^2)^2}\right]$$

$$\boxed{L[t \cos at] = \left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]}$$

7. Integral of Laplace Transform:

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\, ds$$

Proof:

$$\int_s^\infty L[f(t)]\, ds = \int_s^\infty \left[\int_0^\infty f(t) e^{-st}\, dt\right] ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\, ds\right] dt$$

$$= \int_0^\infty \left[f(t) \frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty \frac{f(t)}{t} e^{-st}\, dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

or, $\boxed{L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\, ds}$

example: $L\left[e^{-4t} \frac{\sin 3t}{t}\right]$

$L[\sin 3t] = \frac{3}{s^2+9}$ $\left[\because L[\sin at] = \frac{a}{s^2+a^2}\right]$

$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty \frac{3}{3^2+s^2} ds$ $\left[\because L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)] ds\right]$

$= \frac{3}{3} \tan^{-1} \frac{s}{3} \Big|_s^\infty$ $\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}\right]$

$= \frac{\pi}{2} - \tan^{-1} \frac{s}{3}$

$L\left[\frac{\sin 3t}{t}\right] = \cot^{-1} \frac{s}{3}$

Now, $L\left[e^{-4t} \frac{\sin 3t}{t}\right] = \cot^{-1} \frac{s+4}{3}$ (Ans).

These topics were ~~included~~ in the original syllabus, but has been excluded from the truncated syllabus.

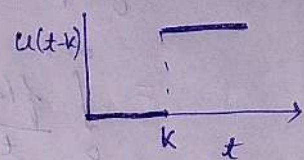
Since these are important with respect ~~in regards~~ to Laplace Transform, I am providing this pdf for your reference. Incase you need a class, do let me know. These might be helpfull for other competitive exams.

Class III

Heaviside Step Function / Unit Step Function:

Heaviside step function is defined as

$$u(t-K) = \begin{cases} 0 & t < K \\ 1 & t > K \end{cases}$$



Because of its utility in describing discontinuous signals pulses, its ~~LT~~ Laplace Transform occurs frequently.

Also with the help of unit step function, the inverse L.transform of certain ~~fe~~ functions can be found easily.

* L.T of Heaviside Step Function:

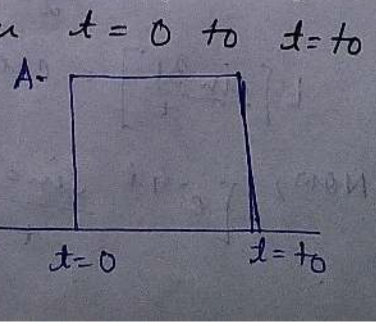
$$L[u(t-K)] = \int_0^\infty e^{-st}\,[u(t-K)]\,dt$$

$$= \int_K^\infty e^{-st}\,dt$$

$$= \left.\frac{e^{-st}}{-s}\right|_K^\infty$$

$$= \frac{e^{-sK}}{s} \quad \text{— (1)}$$

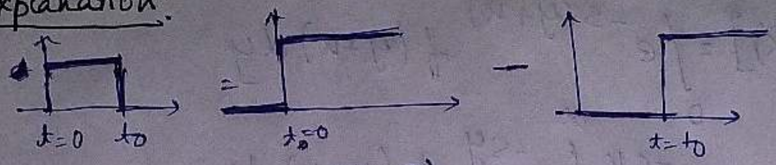
Q. Find the L.T of a square pulse $f(t)$ of height A that is on from $t=0$ to $t=t_0$.



Soln:
Using Heaviside function, the square pulse can be written as,

$$f(t) = A[u(t-0) - u(t-t_0)]$$

Explanation:



$$= u(t-0) \quad - \quad u(t-t_0)$$

$\therefore$, the transform is given by,

$$L[f(t)] = A\left[L\{u(t-0)\} - L\{u(t-t_0)\}\right]$$

or, $$F(s) = A\left[\frac{1}{s} - \frac{e^{-st_0}}{s}\right] \text{ (Ans).} \quad \text{(using 1)}$$

Second Shifting Theorem

If $L[f(t)] = F(s)$,

then, $L[f(t-K)\,u(t-K)] = e^{-sK} F(s)$

Proof:

$$L[f(t-K)\,u(t-K)] = \int_0^\infty e^{-st}\left[f(t-K)\,u(t-K)\right]dt$$

$$= \int_K^\infty e^{-st} f(t-K)\,dt$$

Let $u = t-K$

$du = dt$.

$$\therefore L[f(t-K)\,u(t-K)] = \int_0^\infty e^{-s(u+K)} f(u)\,du$$

$$= e^{-sK}\int_0^\infty e^{-su} f(u)\,du$$

$$= e^{-sK} F(s) \quad \text{(proved)}$$

** Prove: $L[f(t)\,u(t-K)] = e^{-sK} L[f(t+K)]$

Proof: $$L[f(t)\,u(t-K)] = \int_0^\infty e^{-st} f(t)\,u(t-K)\,dt$$

$$= \int_K^\infty e^{-st} f(t)\,dt$$

Let $y = t-K$

$$\therefore, L[f(t)u(t-K)] = \int_0^\infty e^{-s(y+K)} f(y+K)\,dy$$

$$= e^{-sK}\int_0^\infty e^{-sy} f(y+K)\,dy$$

$$= e^{-sK}\int_0^\infty e^{-st} f(t+K)\,dt$$

$$= e^{-sK} L[f(t+K)] \quad \text{(proved)}$$

Q. Express the function as unit step function and hence find the L.T.

$$f(t) = \begin{cases} t-1 & 1<t<2 \\ 3-t & 2<t<3 \end{cases}$$

$\Rightarrow (t-1)[u(t-1) - u(t-2)]$

$\Rightarrow 3-t\,[u(t-2) - u(t-3)]$

Soln:

$$f(t) = (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$

$$= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$$

Taking L.T,

$$L[f(t)] = e^{-s}L(t) - 2e^{-2s}L(t) + e^{-3s}L(t)$$

$$= \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2} \quad \text{(Ans)}$$

H.W. Find the L.T of $t^2u(t-3)$

** Laplace Transform of Dirac Delta Function:

Soln:

$$L[\delta(t-t_0)] = \int_0^\infty e^{-st}\delta(t-t_0)\,dt$$

$$= e^{-st_0}.1 \quad \text{(Ans).}$$

*

Question: ~~Find~~ Evalute $\int_0^\infty e^{-3t}\delta(t-4)\,dt$

Soln: $\int_0^\infty e^{-3t}\delta(t-4)\,dt = e^{-3\times 4}$

$= e^{12}$ (Ans).

Class : 4

* **Laplace Transform of Periodic Function:**

If $f(t)$ be a periodic function of period T, $[f(t) = f(t+T)]$

then

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

Proof:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \dots$$

substituting $u = t - T$ and $u = t - 2T$ ~~and so on~~ in the first and second integral respectively, and so on, we get

$$L[f(t)] = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \dots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-s2T}\int_0^T e^{-su} f(u)\,du + \dots$$

$[\because f(t) = f(t+T) = f(t+2T) = \dots]$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + \dots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + \dots\right] \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}} \qquad \left[\because \frac{1}{1-a} = 1 + a + a^2 + \dots\right]$$

(proved)

Question: Find the LT of the periodic func

$$f(t) = e^{t} \qquad 0<t<2\pi$$

Soln:

$$L[f(t)] = \frac{\int_0^{2\pi} e^{-st} e^{t} dt}{1-e^{-s2\pi}}$$

$$= \frac{1}{1-e^{2\pi s}} \left[\int_0^{2\pi} e^{t(1-s)} dt \right]$$

$$= \frac{1}{1-e^{2\pi s}} \left[\frac{e^{(1-s)t}}{1-s} \Big|_0^{2\pi} \right]$$

$$L[f(t)] = \frac{e^{(1-s)2\pi} - 1}{(1-e^{2\pi s})(1-s)} \quad \text{(Ans)}$$

Question: Find the L.T of half wave rectified sine wave given by:

$$f(t) = \begin{cases} \sin \omega t & 0<t<\pi/\omega \\ 0 & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Soln:

$$L[f(t)] = \frac{1}{1-e^{-s.2\pi/\omega}} \left[\int_0^{2\pi/\omega} f(t)\, e^{-st} dt \right]$$

$$= \frac{1}{1-e^{2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt \right]$$

** $I = \int e^{-st} \sin \omega t \, dt$

$$= \sin \omega t \frac{e^{-st}}{-s} - \int \omega \cos \omega t \frac{e^{-st}}{-s} dt + c$$

$$= -\frac{e^{-st} \sin \omega t}{s} + \frac{\omega}{s} \left[\cos \omega t \frac{e^{-st}}{-s} - \int (-\omega \sin \omega t) \frac{e^{-st}}{-s} dt \right] + c$$

$$I = \frac{-e^{-st} \sin \omega t}{s} - \frac{\omega}{s^2} \cos \omega t \, e^{-st} - \frac{\omega^2}{s^2} I + c$$

or, $\left(1+\frac{\omega^2}{s^2}\right) I = \frac{-e^{-st}}{s^2} \left[s \sin \omega t + \omega \cos \omega t \right] + c$

or, $I = \frac{-e^{-st}}{s^2+\omega^2} \left[s \sin \omega t + \omega \cos \omega t \right] + c$

$$\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt = -\frac{1}{s^2+\omega^2}\left[e^{-s\pi/\omega}\,\omega(-1) - \omega.1\right]$$

$$= \frac{\omega}{s^2+\omega^2}\left[e^{-s\pi/\omega}+1\right].$$

$$\therefore L[f(t)] = \frac{\omega\left[e^{-s\pi/\omega}+1\right]}{s^2+\omega^2} \times \frac{1}{1-e^{-2\pi s/\omega}}$$

$$= \frac{\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})} \quad \text{(Ans)}$$

Imp

** Convolution Theorem:

If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$

then,

$$L[f(t) * g(t)] = F(s)\,G(s)$$

or equivalently,

$$L^{-1}[F(s)\,G(s)] = f(t) * g(t)$$

where $f(t) * g(t)$ is called the convolution of $f(t)$ and $g(t)$ and is defined by the integral,

$$f(t) * g(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau.$$

Proof:

We have by definition,

$$L[f(t) * g(t)] = \int_0^\infty e^{-st}\,dt \int_0^t f(t-\tau)\,g(\tau)\,d\tau. \quad -(1)$$

where the region of integration in the τ-t plane is shown in fig 1.

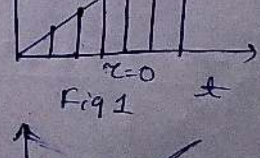


Fig 1

The integration is first performed wrt τ from $\tau=0$ to $\tau=t$ of the vertical strip and then from $t=0$ to ∞ by moving the vertical strip from $t=0$ outwards to cover the whole region under the ~~line~~ line $\tau = t$.

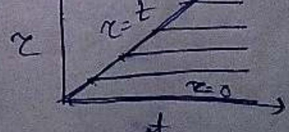


Fig 2

We now change the order of integration so that we integrate along the horizontal strip from $t=\tau$ to $t=\infty$ and then from $\tau=0$ to ∞ by moving the horizontal strip vertically from $\tau=0$ upwards.

$\therefore$ eq 1 becomes,

$$L[f(t)*g(t)] = \int_0^\infty g(\tau)\,d\tau \int_{t=\tau}^{t=\infty} e^{-st} f(t-\tau)\,dt$$

Let $t-\tau = z$

$$\therefore L[f(t)*g(t)] = \int_0^\infty g(\tau)\,d\tau \int_0^\infty e^{-s(\tau+z)} f(z)\,dz$$

$$= \int_0^\infty e^{-s\tau} g(\tau)\,d\tau \int_0^\infty e^{-sz} f(z)\,dz$$

$$= \cancel{G(s)}\ G(s)\,F(s) \qquad \text{(proved)}$$

Question: ① $t*e^{at}$

② $t*t*t$

Soln: ① $t*e^{at} = \int_0^t \tau e^{a(t-\tau)}\,d\tau = e^{at}\int_0^t \tau e^{-a\tau}\,d\tau = \dots$

$$= \frac{1}{a^2}\left(e^{at}-at-1\right) \text{ (Ans)}$$

② $t*t*t = (t*t)*t$

$$= \left[\int_0^t (t-\tau)\tau\,d\tau\right]*t$$

$$= \left[\int_0^t (t\tau-\tau^2)\,d\tau\right]*t = \dots$$

$$= \frac{t^3}{6}*t$$

$$= \frac{1}{6}\int_0^t t^3(t-\tau)\,d\tau = \dots = \frac{t^5}{5!}$$