Linear Vector space

Preliminary Concepts:

* **Binary Operation:**
A binary operation on a set X is a function $F: X \times X \rightarrow X$.

Example: If $\mathbb{Z}$ denote the set of integers, then are 2 binary operations on $\mathbb{Z}$, addition and multiplication. defined by $F_+(m,n) = m+n$ and $F_\cdot(m,n) = mn$
* Division is not a binary operation of $\mathbb{Z}$

** **Group:**
A group is a set G with a binary operation written $(x,y) \rightarrow xy$ such that:

(i) $(xy)z = x(yz)$ for all $x, y, z \in G$ (Associative Law)

(ii) G contains an identity element 1 such that $1x = x1 = x$ for all $x \in G$.

(iii) If $x \in G$, then there exists $y \in G$ such that $xy = 1$. In this case, we say that every element of G has a right inverse.

Example: The integers $\mathbb{Z}$, form a group under addition. ~~$x, y \in \mathbb{Z}$, x~~

(1) Addition is associative

(2) Let $x \in \mathbb{Z}$

$x + 0 = 0 + x = x$ $\therefore$ 0 is the additive identity.

(3) $x + (-x) = 0$

$\therefore$ right inverse of each element exists.

$\therefore$ Integers form a group under addition.

* **Abelian Group:** A group G is said to be abelian if and only if, for all $x, y \in G$, $xy = yx$ (commutative)

H.W: Show that ~~the group~~ $G = \{1, -1\}$ under multiplication in an abelian group.

Some basic properties of Groups:

1) In every group, there is exactly one identity element 1. (~~unique~~

2) If 'y' is the right inverse of 'x', then, $xy = yx = 1$ (ie: each $x \in G$ has a two sided inverse)

3) Each two sided inverse is unique.

Proof 1: Uniqueness of Identity.

Let 1 and 1' be the identity element of a group G.

Then, $1 = 1 \cdot 1' = 1'$

$\Rightarrow 1 = 1' \Rightarrow$ Identity is unique

Proof 2: Left inverse = right inverse

Let, $x, y, z \in G$

let y be the right inverse of x

$\Rightarrow xy = 1$ — (a)

let z be the right inverse of y

$\Rightarrow yz = 1$ — (b)

** Note
[we have to prove $xy = yx = 1$
$\therefore$ if we can prove $x = z$, our purpose gets fullfilled]

Now,

$x = x1$ ~~or x(yz)~~ [$\because$ 1 is identity]

$= x(yz)$ [$\because yz = 1$]

$= (xy)z$ [associative property of group]

$= 1z$ [$\because xy = 1$]

$x = z$ [$1 \rightarrow$ identity]

$\therefore$ left inverse = right inverse.

$\therefore xy = yx = 1$

Proof 3: Uniqueness of each 2 sided inverse:

(H.W)

[Hint: consider two different 2 sided inverse, say y and z of x
$\Rightarrow xy = yx = 1$
& $xz = zx = 1$]

Question: Prove that for all $x, y \in G$,
$(xy)^{-1} = y^{-1}x^{-1}$

Soln:
We know, $(xy)^{-1}xy = 1$ — (a)

Now, $(y^{-1}x^{-1})(xy)$
$= y^{-1}(x^{-1}x)y$ $\quad (x^{-1}x = 1)$
$= y^{-1}y$
$\Rightarrow (y^{-1}x^{-1})(xy) = 1$ — (b)

$\Rightarrow$ comparing (a) and (b), $(xy)^{-1} = y^{-1}x^{-1}$
(proved)

** Fields

A field is a set $\mathbb{F}$ with 2 binary operations, addition ($F_+(a,b) = a+b$) and multiplication ($F.(a,b) = ab$).
First requirement is,
$\mathbb{F}$ be an abelian group under addition
Second,
If 0 denotes the additive identity of $\mathbb{F}$, then $F^* = F - \{0\}$ is an abelian group under multiplication. F^* contains a multiplicative identity 1 such that $0 \neq 1$
In addition, addition and multiplication are related by distributive law.
$a(b+c) = ab + ac$ for all $a, b, c \in \mathbb{F}$

eg: The rational no., real no., complex no.s form fields.

Question: Prove that, for fields, $a0 = 0$ where $a, 0 \in \mathbb{F}$ such that 0 is the additive identity.

Soln: $0 = a0 + (-a0)$ [$\because$ ~~applying~~ $-a0$ is the additive inverse of $a0$]
$= a(0+0) + (-a0)$
$= a0 + a0 + (-a0)$ [$\because$ 0 is the additive identity]
$= a0 + [a0 + (-a0)]$ [associativity under addition]
$\boxed{0 = a0}$ [$\because a0 + (-a0) = 0$]

** If F be a field such that $a, b \in F$ satisfying $ab = 0$, then either $a = 0$ or $b = 0$.

Proof: Let $a \neq 0$

then,

$0 = a^{-1}0$ — $a^{-1} \to$ multiplicative inverse of a $\Rightarrow aa^{-1} = 1$

$= a^{-1}(ab)$ — [∵ $ab = 0$ (given)]

$= (a^{-1}a)b$

$= 1b$

$0 = b$

$\Rightarrow b = 0$.

Similarly if $b \neq 0$, it can be proved $a = 0$.

H.W Show that additive and multiplicative inverse are unique for each elements in a Field:

[Hint: if, $a + b = 0$ $\Rightarrow$ b is the additive inverse of a

$0 \to$ additive identity

if, $a \cdot b = 1$ $\Rightarrow$ b is the multiplicative inverse of a

$1 \to$ multiplicative identity

Proof will be same as groups
* take 2 different inverses in each case (b and b') and ~~equate~~ the finally prove them equal]

H.W. Prove the additive and multiplicative identity ~~inverse~~ are unique for a field

** Show that in a field, the additive identity cannot have a multiplicative inverse

Sol: Let the additive and multiplicative identities be 0 and 1 respectively for a field.

For the additive identity to have a multiplicative inverse, we must have,

$0 \cdot a = 1$ —(i) where $a \in \mathbb{F}$

now; $0 = a0 + (-a0) = a(0+0) + (-a0) = a0$ (as already proved above)

or, $a \cdot 0 = 0$

$\Rightarrow a \cdot 0 \neq 1$ (Hence, proved . . .)

** Vector space:

A (real) vector space is a collection V of vectors together with two binary operations, addition of vectors (+) and scalar multiplication of a vector by a real number (·), satisfying the following axioms:

Let v_1, v_2, v_3 be any vectors in V and $\alpha_1, \alpha_2, \alpha_3$ be any (real numbers) scalars.

Abelian group under addition

1. [A1] Addition is commutative: $v_1+v_2=v_2+v_1$
2. [A2] " " associative: $(v_1+v_2)+v_3 = v_1+(v_2+v_3)$
3. [A3] There exists a zero vector 0, such that $v+0=v$
4. [A4] Every vector v has an additive inverse $-v$ such that $v+(-v)=0$
5. [M1] Scalar multiplication is consistent with regular multiplication: $\alpha_1(\alpha_2 v) = (\alpha_1\alpha_2).v$
6. [M2] Addition of scalars distribute: $(\alpha_1+\alpha_2)v = \alpha_1 . v + \alpha_2 . v$
7. [M3] Addition of vectors distribute: $\alpha.(v_1+v_2) = \alpha.v_1 + \alpha.v_2$
8. [M4] The scalar 1 acts like the identity on vectors: $1.v = v$.

** The most general notion of vector space involves scalar from a field.

** Lets see whether the vectors in ~~$\mathbb{R}$~~ $\mathbb{R}^2$ ~~are~~ form a vector space or not:

[Notation: We will write the vector as ordered tuple of coordinates. The n-D vector from origin to $(a_1, a_2 \ldots a_n)$ is written a $v = \langle a_1, a_2 \ldots, a_n\rangle$]

Let the vectors of $\mathbb{R}^2$ be written as $\langle x, y\rangle$ and scalar multiplication defined by $\alpha \cdot \langle x,y\rangle = \langle \alpha x, \alpha y\rangle$

* $A[1]$: $\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1+x_2, y_1+y_2\rangle = \langle x_2, y_2\rangle + \langle x_1, y_1\rangle$.

* $A[2]$: $(\langle x_1, y_1\rangle + \langle x_2, y_2\rangle) + \langle x_3, y_3\rangle = \langle x_1+x_2+x_3, y_1+y_2+y_3\rangle$
$= \langle x_1, y_1\rangle + (\langle x_2, y_2\rangle + \langle x_3, y_3\rangle)$

* $A[3]$: The zero vector $\langle 0,0\rangle$: $\langle x,y\rangle + \langle 0,0\rangle = \langle x,y\rangle$

* $A[4]$: The additive inverse of $\langle x,y\rangle$ is $\langle -x,-y\rangle$:
$\langle x,y\rangle + \langle -x,-y\rangle = \langle 0,0\rangle$

* $[M1]$: $\alpha_1 \cdot (\alpha_2 \cdot \langle x,y\rangle) = \langle \alpha_1\alpha_2 x, \alpha_1\alpha_2 y\rangle = (\alpha_1\alpha_2)\cdot\langle x,y\rangle$

* $[M2]$: $(\alpha_1+\alpha_2)\langle x,y\rangle = \langle(\alpha_1+\alpha_2)x, (\alpha_1+\alpha_2)y\rangle$
$= \alpha_1 \cdot \langle x,y\rangle + \alpha_2 \cdot \langle x,y\rangle$.

* $M[3]$: $\alpha(\langle x_1,y_1\rangle + \langle x_2,y_2\rangle) = \langle \alpha(x_1+x_2), \alpha(y_1+y_2)\rangle$
$= \alpha \cdot \langle x_1,y_1\rangle + \alpha \cdot \langle x_2,y_2\rangle$

* $M[4]$: Scalar 1 exists.
ie $1 \cdot \langle x,y\rangle = \langle x,y\rangle$.

$\therefore$ the set of vectors in $\mathbb{R}^2$ forms a vector space.

#HW

Assignment: (3 marks)

(1) Show that the set of $m \times n$ matrices for any m and n any n forms a vector space.

Subspaces:

A subspace W of a vector space V is a subset of vector space V, which under the same addition and scalar multiplication operations as V, is itself a vector space.

Eg: Show that the set of diagonal 2X2 matrices is a subspace of the vector space of all 2x2 matrices.

* The form of matrix accn to question is $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

① $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix} \Rightarrow$ diagonal matrix

$p \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} pa & 0 \\ 0 & pb \end{bmatrix} \Rightarrow$ diagonal matrix

well defined operation. (closed under addition & scalar mult.)

② commutative, associative holds

③ $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (additive identity exists)

④ $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (additive inverse exists)

⑤ $p \cdot q \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} pqa & 0 \\ 0 & pqb \end{bmatrix} = pq \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ [Scalar multiplication consistent with regular mult]

⑥ M2, M3, M4 holds

∴ Subspace proved

It is not required to prove all the properties. The ~~ones given Below the ... is sufficient~~

* Subspace Criterion:

A ~~st~~ subset W of a vector space V is a subspace of V if and only if W has the following properties:

① ~~[...]~~ W contains the zero vector of V. (additive identity)

② W is closed under addition.

③ " " " scalar multiplication.

(27)

Q) Determine whether the set of vectors of the form $\langle t, t^2 \rangle$ forms a supspace of $\mathbb{R}^2$.

Soln:

① $\langle t, t^2 \rangle + \langle 0, 0 \rangle = \langle t, t^2 \rangle$ where $\langle 0,0 \rangle$ is of the same form (t, t^2) such that $t=0$

$\langle 0, 0 \rangle$ =

② $\langle t_1, t_1^2 \rangle + \langle t_2, t_2^2 \rangle = \langle t_1 + t_2, t_1^2 + t_2^2 \rangle$

$\Rightarrow$ it is not closed under addition.

$\therefore \langle t, t^2 \rangle$ is not a subspace of $\mathbb{R}^2$.

H.W ① Determine whether the set of vectors of the form $\langle s, t, 0 \rangle$ form a subspace of $\mathbb{R}^3$

② Determine whether the set of 2X2 matrices of trace zero is a subspace of the space of 2X2 matrices.

Linear combination and Span:

① Given a set $v_1, v_2 \ldots v_n$ of vectors in a vector space V, vector W in V is a linear combination of $v_1, v_2, \ldots v_n$ if there exists scalars $a_1, a_2 \ldots a_n$ such that,

$$W = a_1 V_1 + a_2 V_2 + \ldots + a_n V_2.$$

example: in $\mathbb{R}^2$, $\langle 1, 5 \rangle$ is a linear combination of $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ such that, $\langle 1, 5 \rangle = 1 \cdot \langle 1, 0 \rangle + 5 \langle 0, 1 \rangle$

② The span of a collection of vectors $V_1, V_2 \ldots V_n$ in V, denoted as span $(V_1, V_2 \ldots V_n)$ is given by the set W of all vectors which are linear combinations of $V_1, V_2 \ldots V_n$.

Explicitly, span is the set of vectors of the form $a_1 V_1 + a_2 V_2 + \ldots a_n V_n$ for some scalars $a_1, a_2 \ldots a_n$.

eg: The span of vectors $\langle 1, 0, 0 \rangle$ and $\langle 0, 1, 0 \rangle$ in $\mathbb{R}^3$ is the set of vectors of the form,

$$a \cdot \langle 1, 0, 0 \rangle + b \langle 0, 1, 0 \rangle = \langle a, b, 0 \rangle.$$

8) Determine whether the vectors $\langle 2,3,3\rangle$ and $\langle 4,-1,3\rangle$ are in span $\langle v,w\rangle$, where $v=\langle 1,-1,2\rangle$ and $w=\langle 2,1,-1\rangle$

Soln:

$\langle 2,3,3\rangle$:

we must determine whether is is possible to write,

$\langle 2,3,3\rangle = a\langle 1,-1,2\rangle + b\langle 2,1,-1\rangle$ for some a and b.

This gives the equations:

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$a + 2b = 2$

$-a + b = 3$

$2a - b = 3$

Applying Gauss Elimination method,

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ -1 & 1 & 3 \\ 2 & -1 & 3 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 5 \\ 0 & -5 & -1 \end{array}\right] \xrightarrow{R_3+\frac{5}{3}R_2} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 22/3 \end{array}\right]$$

$-1+\frac{5}{3}\times 5$

$-1 + \frac{25}{3}$

$\frac{-3+25}{3}$

As the entire 3rd row is zero, it is an inconsistent system. $0 = 22/3$?! inconsistent

Thus, $\langle 2,3,3\rangle$ is not in the span.

H.W ~~determine for~~ Prove that $\langle 4,-1,3\rangle$ is in the span.

(Mark = 3)

** ~~H.W~~ Assignment: Show that for any vectors $V_1, V_2 \ldots V_n$

(2) in V, the set span $(V_1, V_2 \ldots V_n)$ is a subspace of V.

** Given a vector space V, if the span of the vectors $V_1, V_2 \ldots V_n$ is all of V, we say ~~span or generate~~ the vectors $V_1, V_2 \ldots V_n$ ~~the~~ form a spanning set (or generating set) of V.

eg: $\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle$ span $\mathbb{R}^3$

as any vector $\langle a,b,c\rangle$ is a linear combination of them. ie:

$\langle a,b,c\rangle = a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle$

3(8) Q) Determine whether the vectors $\langle 2,3,3\rangle$ and $\langle 4,-1,3\rangle$ are in span $\langle v,w\rangle$, where $v=\langle 1,-1,2\rangle$ and $w=\langle 2,1,-1\rangle$

Soln:

$\langle 2,3,3\rangle$:

we must determine whether is is possible to write,

$\langle 2,3,3\rangle = a\langle 1,-1,2\rangle + b\langle 2,1,-1\rangle$ for some a and b.

This gives the equations!

$\Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

$a+2b=2$

$-a+b=3$

$2a-b=3$

Applying Gauss Elimination method,

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ -1 & 1 & 3 \\ 2 & -1 & 3 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 5 \\ 0 & -5 & -1 \end{array}\right] \xrightarrow{R_3+\frac{5}{3}R_2} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 22/3 \end{array}\right]$$

As the entire 3rd row is zero, it is an inconsistent system. $0=22/3$?! inconsistent

Thus, $\langle 2,3,3\rangle$ is not in the span.

~~determine for~~ Prove that $\langle 4,-1,3\rangle$ is in the span.

Assignment (Mark = 3)

** ~~HW~~ Show that for any vectors $v_1, v_2 \ldots v_n$ (2) in V, the set span $(v_1, v_2 \ldots v_n)$ is a subspace of V.

** Given a vector space V, if the span of the vectors $v_1, v_2 \ldots v_n$ is all of V, we say the vectors $v_1, v_2 \ldots v_n$ ~~span or generate~~ ~~the~~ form a spanning set (or generating set) of V.

eg: $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$, $\langle 0,0,1\rangle$ span $\mathbb{R}^3$

as any vector $\langle a,b,c\rangle$ is a linear combination of them. ie:

$\langle a,b,c\rangle = a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle$

Span (continuation)

Q) Determine whether $\langle 1,-1,3\rangle$, $\langle 1,0,1\rangle$, $\langle -1,2,-5\rangle$ span $\mathbb{R}^3$

Soln: vector form in $\mathbb{R}^3 \Rightarrow$ ~~$\langle a,b$~~ $\langle p,q,r\rangle$

For the given vectors to span $\mathbb{R}^3$,

$$a\langle 1,-1,3\rangle + b\langle 1,0,1\rangle + c\langle -1,2,-5\rangle = \langle p,q,r\rangle$$

must exist.

Therefore we get the equations:

$$a+b-c = p$$
$$-a+2c = q$$
$$3a+b-5c = r$$

~~or, we can write,~~

$$\cancel{\begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \Rightarrow Ma = 1}$$

~~Using Gauss Elimination:~~

Using Gauss Elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & p \\ -1 & 0 & 2 & q \\ 3 & 1 & -5 & r \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & p \\ 0 & 1 & 1 & p+q \\ 0 & -2 & -2 & r-3p \end{array}\right] \xrightarrow{R_3 = R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & p \\ 0 & 1 & 1 & p+q \\ 0 & 0 & 0 & r-p+2q \end{array}\right]$$

This ~~is~~ gives inconsistent result. hence they do not span $\mathbb{R}^3$

** From the above analysis, we can ~~write~~, ~~say~~, write

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \text{ or } \boxed{Mx = b}$$

where $M \rightarrow$ matrix, ~~with each~~ whose columns are the given vectors $v_1, v_2 \ldots$

$x \rightarrow$ solution vector

$b \rightarrow$ vector

i.e. A collection of vectors $v_1, v_2 \ldots$ in $\mathbb{R}^n$ will span $\mathbb{R}^n$ if and only if, for every vector ~~b~~ b, there is at least 1 vector x satisfying the matrix equation $Mx = b$.

Such a solution exists for any b if and only if M has rank n.

($\mathbb{R}^n$)

9(2)

* Linear Independence and Linear Dependence

A finite set of vectors $v_1, v_2 \ldots v_n$ is linearly independent if $a_1 v_1 + a_2 v_2 \ldots + a_n v_n = 0$ Implies $a_1 = a_2 = \ldots a_n = 0$.

Otherwise, the collection is linearly dependent.

[a set is linearly dependent if one ~~the~~ of the vector is a linear combination of the others]

eg: $\langle 0,1,0 \rangle, \langle 1,0,0 \rangle, \langle 0,0,1 \rangle$ are linearly independent

Q) Determine whether ~~⟨p~~ the vectors $\langle 1,1,0 \rangle$, $\langle 0,2,1 \rangle$ in $\mathbb{R}^3$ are linearly dependent or independent.

Soln: Suppose there are a and b such that

$$a\langle 1,1,0 \rangle + b\langle 0,2,1 \rangle = \langle 0,0,0 \rangle.$$

comparing,

$a = 0$
$a + 2b = 0$
$b = 0$

These 3 eqns. can only be true if $a = b = 0$.

$\therefore$ these vectors are linearly independent

H.W Determine whether $\langle 1,0,2,2 \rangle, \langle 2,-2,3,0 \rangle$, $\langle 0,3,1,3 \rangle$ and $\langle 0,4,1,2 \rangle$ in $\mathbb{R}^4$ are linearly dependent or independent

** Thus, we can generalise the above findings as,

A collection of k vectors $v_1, v_2 \ldots v_n$ in $\mathbb{R}^n$ is linearly dependent if and only if, there is a non zero vector 'x' satisfying the matrix eqn $Mx = 0$, where M is the matrix whose columns are $v_1, v_2 \ldots v_n$.

** The vectors will be independent if $x = 0$.

4(3)

** **Basis:** A linearly independent set of vectors which spans V (a vector space) is called a basis for V.

eg: Show that vectors $\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle$ form a basis for $\mathbb{R}^3$

Soln:

Form of vector in $\mathbb{R}^3 \Rightarrow \langle a,b,c\rangle$

$$\langle a,b,c\rangle = a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle$$

Thus any vector can be written as linear combination of the given vectors

also,

$$\langle 0,0,0\rangle = a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle$$

is only possible when $a=b=c=0$

$\therefore$ They are linearly independent.

$\therefore$ The given vectors form a basis for $\mathbb{R}^3$

H.W. ASSIGNMENT (Marks: 5)

(3) Q: Show that $\langle 1,1,1\rangle, \langle 2,-1,1\rangle, \langle 1,2,1\rangle$ form a basis of $\mathbb{R}^3$

[Hint: Check the linear independence ~~and the existence of~~ $\cancel{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}$ ~~vector~~ criteria (ie: $Mx=0$ such that $x=0$) and spanning set criteria ~~such that~~ $Mx=b$ such that x exists for each b. Use Gauss Elimination for calculations]

** **Dimension:**

If V is a vector space, the number of elements in any basis of V is called the dimension of V and is denoted by $\dim(V)$.

IMP. PROPOSITION:

If V is an n-dimensional vector space, then any set of fewer than 'n' vectors cannot span V, and any set of more than n vectors is linearly dependent.

Change of Basis

** Standard basis is the simplest basis of the space of all n-dimensional vectors. It is made up of vectors that have one entry equal to 1 and the remaining n-1 entries equal to 0.

Defn: Let S be the space of all n-dimensional vectors. Denote by e_k, a vector whose k-th entry is equal to 1 and whose ~~re~~ remaining n-1 ~~ee~~ entries are equal to 0. Then, the set of n vectors is,

$$\{e_1, e_2 \ldots e_k\}$$

is called the standard basis of S.

** Ordered basis is a list rather than a set. ie: the order of vectors in a ordered basis matters.

** Components:

If $U = u_1, u_2 \ldots u_n$ is an ordered ~~basis~~ basis for V and v is a vector in V, then there's a unique list of scalars $c_1, c_2 \ldots c_n$ such that:

$$v = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$$

These ~~ar~~ $[c_i]$ are called the components of v relative to the ordered basis U.

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}_U = [v]_U$$ represents the component vector of v w.r.t basis U.

eg: ~~Say~~ Say in $\mathbb{R}^3$, we have a vector,

$$\vec{a} = 5\hat{i} + 3\hat{j} + 3\hat{k}$$

- This can be written as

$$a = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$$

Careful

where $\hat{i}, \hat{j}, \hat{k}$ → form the standard basis

$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\hat{j} =$

Just for understanding **

Notation

5(2)

A vector, say A in the $\mathbb{R}^3$ vector space can be written using the following notations.

$|A\rangle$ Bra-Ket / Dirac notation

say, $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$ —— (1)

Here, $\hat{i}, \hat{j}, \hat{k} \rightarrow$ form the ordered standard basis

such that, $\hat{i}$ can be equivalently written as,

$$\hat{i} = |e_1\rangle = \langle 1,0,0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Dirac notation/ Bra Ket notation.

$$\hat{j} = |e_2\rangle = \langle 0,1,0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\hat{k} = |e_3\rangle = \langle 0,0,1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Similarly,

$A = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow$ component vector of A w.r.t standard basis.

~~In matrix notation, (1) can be wr~~

eq. (1) can be written in a number of ways:

(1) $\langle a,b,c\rangle = a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle$

(2) $|A\rangle = a|e_1\rangle + b|e_2\rangle + c|e_3\rangle$

where $|e_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $|e_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $|e_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(3) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Big\}$ equivalent statements

CHANGE OF BASIS

Theorem

Let $S = \{v_1, v_2, \dots v_n\}$ be a basis for a vector space V and let v be a vector in V. Then

$$[v]_S = A^{-1} v$$

where A is the matrix whose column vectors are $\{v_1, v_2, \dots v_n\}$

~~Example~~: ~~H.W~~ Assignment: (3).

Let $S = \{\langle 1,3,4\rangle, \langle 2,-1,1\rangle, \langle 1,0,2\rangle\}$

be a basis for $\mathbb{R}^3$ and let

$$[v]_S = (2,3,-1)$$

Find the coordinates w.r.t the standard basis.

~~Soln~~

Example:

Consider $S = \{\underset{v_1}{\langle 1,2\rangle}, \underset{v_2}{\langle 4,7\rangle}\}$ of $\mathbb{R}^2$ is a basis.

Let a vector $v = (5,8)$ presented in standard basis. Find the coordinates of v in the basis S ie $[v]_S$.

Soln: Let $[v]_S = (a,b)$

$\therefore$ $\langle 5,8\rangle = a\langle 1,2\rangle + b\langle 4,7\rangle$

[more elaborately
~~$5\langle 1,0\rangle + 8\langle 0,1\rangle =$ $a\langle 1,2\rangle + b\langle 4,7\rangle$~~
$5\langle 1,0\rangle + 8\langle 0,1\rangle = a\langle 1,2\rangle + b\langle 4,7\rangle$]

$\therefore$, we get the eqns:

$$a + 4b = 5$$
$$2a + 7b = 8$$

$\Rightarrow$ from here, you can solve for a and b using G. Eli. method.

$\Downarrow$

From here, we get the matrix eqn:

$$\begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$\Rightarrow A[v]_S = v$

$\Rightarrow [v]_S = A^{-1} v$ here $A = ([v_1]\ [v_2]$

J: S(V) ** Change of basis from B (old) to B' (new):

If B and B' are two bases of $\mathbb{R}^n$, with
$B = \{v_1, v_2 \ldots v_n\}$,
then the transition matrix from B coordinates to B' coordinates is given by,

$$M = \left[[v_1]_{B'}, [v_2]_{B'}, \ldots [v_n]_{B'} \right]$$

[ie: the columns of the transition matrix are the coordinate vectors of the old Basis w.r.t the new Basis B'] [Proof! follow video]

Example: Given $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$ are bases in $\mathbb{R}^2$. Given $[x]_B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}_B$

find $[x]_{B'}$

Soln:

Let $[x]_{B'} = \begin{bmatrix} a \\ b \end{bmatrix}$

step for expressing the vectors of old basis w.r.t the new basis.

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ let $[v_1]_{B'} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\Rightarrow \langle 1,2 \rangle = x\langle 3,1 \rangle + y\langle 5,2 \rangle$

$$\Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

~~Usin~~ G. Elim. method,

$$\left[\begin{array}{cc|c} 3 & 5 & 1 \\ 1 & 2 & 2 \end{array}\right] \xrightarrow{R_2 = R_2 - \frac{1}{3}R_1} \left[\begin{array}{cc|c} 3 & 5 & 1 \\ 0 & 1/3 & 5/3 \end{array}\right]$$

$\Rightarrow$ $3x + 5y = 1$ — (1)

$\frac{1}{3} y = \frac{5}{3}$

$\Rightarrow y = 5$

Putting in (1), $x = -8$

$\therefore [v_1]_{B'} = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$ $\Rightarrow$ coordinate of v_1 w.r.t new basis B'.

Similarly, let $[v_2]_{B'} = \begin{bmatrix} x2 \\ y2 \end{bmatrix}$

$\langle -1, 1 \rangle = x2\langle 3,1 \rangle + y2\langle 5,2 \rangle$

Solving, $[v_2]_{B'} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$

$$\therefore [x]_{B'} = \begin{bmatrix} -8 & -7 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -25 \\ 16 \end{bmatrix}_{B'}$$

Linear Transformations:

If V and W are vector spaces, we say a map T from V to W (denoted by $T: V \to W$) is a linear Transformation, if for any vectors v, v_1, v_2 and scalar α, the following properties hold:

- [T1] The map respects addition of vectors:

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

- [T2] The map respects scalar multiplication:

$$T(\alpha \cdot v) = \alpha T(v)$$

Eg:

(1) If $V = W = \mathbb{R}^2$, show that the map T which sends $\langle x, y \rangle$ to $\langle x, x+y \rangle$ is a linear transformation from V to W.

Soln

Let $v = \langle x, y \rangle$ $\quad v_1 = \langle x_1, y_1 \rangle$ $\quad v_2 = \langle x_2, y_2 \rangle$

$$[T1]: T(v_1 + v_2) = T\langle x_1 + x_2, y_1 + y_2 \rangle$$
$$= \langle x_1 + x_2, x_1 + x_2 + y_1 + y_2 \rangle \quad -(1)$$
$$T(v_1) + T(v_2) = T\langle x_1, y_1 \rangle + T\langle x_2, y_2 \rangle$$
$$= \langle x_1, x_1 + y_1 \rangle + \langle x_2, x_2 + y_2 \rangle$$
$$= \langle x_1 + x_2, x_1 + x_2 + y_1 + y_2 \rangle \quad -(2)$$
$$\therefore \text{eqn } 1 = \text{eqn } (2)$$

$$[T2]: T(\alpha \cdot v) = T\langle \alpha x, \alpha y \rangle = \langle \alpha x, \alpha x + \alpha y \rangle$$
$$= \alpha \langle x, x+y \rangle$$
$$= \alpha T(v)$$

Since both the axioms are satisfied, T is a LT.

(2) If V is the vector space of all differentiable functions and W is the vector space of all functions, determine whether the derivative map D sending a function to its derivative is a L.T from V to W.

Soln: $[T1] = D(f_1 + f_2) = (f_1 + f_2)' = f_1' + f_2' = D(f_1) + D(f_2)$

$[T2] = D(\alpha \cdot f) = (\alpha f)' = \alpha \cdot f' = \alpha \cdot D(f)$

[T1], [T2] satisfied.

$\therefore$ LT

(3) If $V = M_{2\times2}(\mathbb{R})$ and $W = \mathbb{R}$ (real), determine whether the trace map is a LT from V.

Soln Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $M_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ $M_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$\left.\begin{aligned}
[T1]: T(M_1+M_2) &= T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) \\
&= a_2+a_1+d_2+d_1 \\
&= (a_2+d_2)+(a_1+d_1) \\
&= T(M_1)+T(M_2) \\
[T2]: T(\alpha M) &= \alpha a+\alpha d \\
&= \alpha(a+d) \\
&= \alpha T(M)
\end{aligned}\right\} \text{proved L.T.}$$

** ASSIGNMENT: (8).

If $V = N_{2\times2}(\mathbb{R})$ and $W = \mathbb{R}$, determine whether the determinant map is a L.T from V to W or not.

** FEW ALGEBRAIC PROPERTIES OF L.T:

(1) Any L.T $T: V \to W$ sends the zero vector of V to to the zero vector of W:

Proof: $v \to$ vector in V.

$0.v = 0_v$ —(1) from basic properties of Linear Vect space.

NOW; $T(0_v) = T(0.v) = \underset{\text{using 1}}{\uparrow}$ $0\,T(v) = 0_W$ $\uparrow$ using [T2] (since scaling any vector of W by 0 gives the zero vector of W)

∴ The, on performing a L.T, the zero vector of the vector space (V), maps to the zero vector of the Image space (W).

(2) ~~Any~~ Any L.T is uniquely defined by its values on a basis of V.

Proof Let V be n-D L.V.S and $v_1, v_2, \ldots v_n$ form a ~~the~~ basis of V.

$\therefore$ Any vector is v can be written as,

$$v = a_1 \cdot v_1 + a_2 \cdot v_2 + \cdots + a_n \cdot v_n$$

Now, on apply L.T, ~~we get~~ (say T), we get

$$T(v) = T(a_1 \cdot v_1 + a_2 v_2 + \cdots + a_n \cdot v_n)$$

$$= T(a_1 \cdot v_1) + T(a_2 \cdot v_2) + \cdots + T(a_n \cdot v_n) \quad \text{using } [T1]$$

$$= a_1 \cdot T(v_1) + a_2 \cdot T(v_2) + \cdots + a_n \cdot T(v_n)$$

So, the value of T on any v (ie: the result value of L.transformation on any vector v) is determined by unique combination of $T(v_1), T(v_2), \ldots T(v_n)$ [ie: value of transformation of the basis vectors]

Imp
**

REPRESENTATION OF L.T BY MATRICES:

Q: If $V = W = \mathbb{R}^2$, then the map T which sends $\langle x, y \rangle$ to $\langle ax + by, cx + dy \rangle$ for any a, b, c, d is a L.T.

Sol.

Solve it as already shown (H.W)

2nd approach Another way to think of this map is as a matrix map.

ie: T sends $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$ This L.Trans is just left mult by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

This makes calculations easier.

$$[T1]: \quad T(v_1 + v_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (v_1 + v_2)$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} v_1 + \begin{pmatrix} a & b \\ c & d \end{pmatrix} v_2 = T(v_1) + T(v_2)$$

$$[T2] = T(\alpha v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\alpha v)$$

$$= \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \alpha T(v)$$

This above finding can be generalized

(**) If $V=\mathbb{R}^m$ (thought as $m\times 1$ matrices) and $W=\mathbb{R}^n$ and A is any $n\times m$ matrix, then the map T sending v to Av is a linear transformation.

Proof

[T1] : $T(v_1+v_2) = A(v_1+v_2) = Av_1+Av_2 = T(v_1)+T(v_2)$

[T2] : $T(\overset{\alpha}{\cancel{v_1+}}v_2) = A\cdot\alpha v = \alpha(A\cdot v)$

$= \alpha T(v)$.

∵ Av is normal matrix multiplication

Matrix Representation of linear transformation:

Let V and W be 2 vector space and $\Gamma=(v_1,v_2,\dots,v_n)$ and $\Omega=(w_1,w_2,\dots,w_m)$ be ordered basis for V and W respectively.

Let $T: V\to W$ be a L.T. Let $v=a_1v_1+a_2v_2+a_3v_3+\dots a_nv_n$ be a vector in V.

Now, for each v_i, $T(v_i)$ is a vector in W

∴ $T(v_i)$'s can be expressed as a linear combn to w_j's.

* ie: $T(v_i)=a_{1i}w_1+a_{2i}w_2+\dots+a_{mi}w_m$

Applying linear transformation on v,

$$
\begin{aligned}
T(v) &= T(a_1v_1+a_2v_2+\dots+a_nv_n)\\
&= \sum_{i=1}^{n} a_i\,T(v_i)\\
&= \sum_{i=1}^{n} a_i\left(\sum_{j=1}^{m} a_{ji}w_j\right)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \cancel{a_{ji}}(a_{ji}a_i)\,w_j\\
&= \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ji}a_i\right)w_j
\end{aligned}
$$

$$
\therefore [T(v)]_\Omega = \begin{bmatrix} \cancel{\sum_{j=1}^{m} a_{1i}a_i} \\ \cancel{\sum_{j=1}^{m} a_{2i}a_i} \\ \vdots \\ \cancel{\Sigma} \end{bmatrix}
\begin{bmatrix} \sum_{i=1}^{n} a_{1i}a_i \\ \sum_{i=1}^{n} a_{2i}a_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}a_i \end{bmatrix}
= \begin{bmatrix} a_{11}a_1+a_{12}a_2+\dots+a_{1n}a_n \\ a_{21}a_1+a_{22}a_2+\dots+a_{2n}a_n \\ \vdots \\ a_{m1}a_1+a_{m2}a_2+\dots+a_{mn}a_n \end{bmatrix}
$$

$$
[T(v)]_\Omega = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}
\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}
$$

$\therefore$ The matrix representation of T is nothing but a matrix of the form

$$M = \left[[V_1]_\Omega , [V_2]_\Omega , [V_3]_\Omega \ldots [V_n]_\Omega \right]$$

ie: The columns are the coordinates of the ~~basis vectors~~ of V on undergoing result of transformation of the basis vectors of V.

Alternative
** Proof (using cong method)

Let the Basis of V be given by $V \Rightarrow \mathbb{R}^n$

$\Gamma = (v_1, v_2 \ldots v_n)$ $W \Rightarrow \mathbb{R}^m$.

and Basis of W be given by,

$\Omega = (\omega_1, \omega_2 \ldots \omega_m)$

$\because$ $T(v_i)$ are vectors of W, $\therefore$ they can be expressed as linear comb n of the ω_j's (i.e; basis).

$$\therefore \quad T(v_1) = a_{11}\omega_1 + a_{21}\omega_2 + \ldots + a_{m1}\omega_m$$

$$T(v_2) = a_{12}\omega_1 + a_{22}\omega_2 + \ldots + a_{m2}\omega_m$$

$$\vdots$$

$$T(v_n) = a_{1n}\omega_1 + a_{2n}\omega_2 + \ldots + a_{mn}\omega_m$$

Now, any vector v in V can be written as

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n$$

Apply L.T,

$$T(v) = T(\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n)$$

$$= \lambda_1 T(v_1) + \lambda_2 T(v_2) + \ldots + \lambda_n T(v_n)$$

$$= \lambda_1 (a_{11}\omega_1 + a_{21}\omega_2 + \ldots + a_{m1}\omega_m) + \lambda_2 (a_{12}\omega_1 + a_{22}\omega_2 + \ldots a_{m2}\omega_m) + \ldots + \lambda_n (a_{1n}\omega_1 + a_{2n}\omega_2 + \ldots + a_{mn}\omega_m)$$

rearranging,

$$T(v) = (\lambda_1 a_{11} + \lambda_2 a_{12} + \ldots \lambda_n a_{1n})\omega_1 + \ldots + (\lambda_1 a_{m1} + \lambda_2 a_{m2} + \ldots \lambda_n a_{mn})\omega_m$$

or,

$$[T(v)]_\Omega = \begin{pmatrix} \lambda_1 a_{11} + \lambda_2 a_{12} + \cdots + \lambda_n a_{1n} \\ \lambda_1 a_{21} + \lambda_2 a_{22} + \cdots + \lambda_n a_{2n} \\ \vdots \\ \lambda_1 a_{m1} + \lambda_2 a_{m2} + \cdots + \lambda_n a_{mn} \end{pmatrix}$$

$$[T(v)]_\Omega = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$$

$$\begin{array}{ccc} [T(v_1)]_\Omega & [T(v_2)]_\Omega & [T(v_n)]_\Omega \end{array}$$

$\hookrightarrow$ $m \times n$ matrix. $\Rightarrow$ unique matrix

Final conclusion =

If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ then the matrix representation (A) of the Linear Transformation $T : V \rightarrow W$ is given by a $m \times n$ matrix whose colums are the linear transformed basis vectors of V.

INNER PRODUCT SPACE

In a general linear vector space, the innerproduct is defined axiomatically as given below:

given 2 vectors $|v\rangle$ and $|w\rangle$

such that $|v\rangle = \sum_{i=1}^{n} u_i |i\rangle$

and $|w\rangle = \sum_{i=1}^{n} w_i |i\rangle$,

$|i\rangle$, $i = 1 \dots n$ represent the basis.

the inner product is given by,

$$\langle v|w\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} u_i^* w_j \langle i|j\rangle \quad \text{—(1)}$$

The inner product obeys the following rules. (or axioms)

1. $\langle v|w\rangle = \langle w|v\rangle^*$ skew symmetry
2. $\langle v|v\rangle \geq 0$ ($= 0$ if and if $|v\rangle = |0\rangle$) (positive semidefiniteness)
3. $\langle v|(|x\rangle + |w\rangle) = \langle v|x\rangle + \langle v|w\rangle$ (additivity)
4. $\langle av|w\rangle = a^* \langle v|w\rangle$,
$\langle v|bw\rangle = b\langle v|w\rangle$.

Notes:

** 2vectors are said to be orthogonal if the inner product vanishes.

** $\langle v|v\rangle^{1/2} = ||v||$ → called the norm or length of vector

** normlized vector having unit norm is called a unit vector.

** Any non zero vector can be can be normalised by dividing it by its length

** An orthonormal basis is a set of basis vectors that are all of unit norm and pair-wise orthogonal.
If $|i\rangle$ and $|j\rangle$ are orthonormal basis vectors.

$$\langle i|j\rangle = \delta_{ij} = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i\neq j \end{cases}$$

∴ eq (1) reduces to,

$$\langle v|w\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} u_i^* w_j \delta_{ij}$$

$$= \sum_i u_i^* w_i$$

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** Assignment (3 + 5)

① Let $|U\rangle = (3-4i)|1\rangle + (5-6i)|2\rangle$

and $|W\rangle = (1-i)|1\rangle + (2-3i)|2\rangle$

be 2 vectors expanded in orthonormal basis $|1\rangle$ and $|2\rangle$.

find: $\langle U|U\rangle$, $\langle W|W\rangle$, $\langle U|W\rangle$

② If A and B be 2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

then show that,

$$\langle A|B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

[Notation: ① $|a\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ (column matrix) (If $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$)

② $\langle a| = (a_1^*, a_2^*, a_3^*)$ (transpose conjugat of a)

③ $\langle a|b\rangle = (a_1^*, a_2^*, \dots, a_n^*)\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$]

[Hint: The standard basis of $\mathbb{R}_{2\times2}$ are:

$$\underset{|1\rangle}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{|2\rangle}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{|3\rangle}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{|4\rangle}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

* express A and B as linear combination of $|1\rangle, |2\rangle, |3\rangle$ and $|4\rangle$ and apply the defn. of inner product (matrix multiplication)]

Example Problem:

Consider a vector, $|U\rangle = \begin{pmatrix} 1+i \\ \sqrt{3}+i \end{pmatrix}$ $i = \sqrt{-1}$ in a certain orthonormal basis.

Expand it is a new orthonormal basis whose components are given by

$$|e_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |e_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Soln: In the new basis, we can write

$$U = a_1|e_1\rangle + a_2|e_2\rangle$$

$$\therefore \langle e_1|U\rangle = a_1\langle e_1|e_1\rangle + a_2\langle e_1|e_2\rangle$$

$$= a_1$$

$$\Rightarrow a_1 = \langle e_1|U\rangle = \frac{1}{\sqrt{2}}(1 \;\; 1)\begin{pmatrix} 1+i \\ \sqrt{3}+i \end{pmatrix} =$$

similarly, $a_2 = \langle e_2|U_2\rangle =$

Do soln