$\oint$ntegration in the Complex Plane

By
Narayan Choudhary
Asst Professor
Ananda Chandra College, Jalpaigur
West Bengal - 735101

CONTENTS :

- Introduction
- Line integrals
- Contour integrals
- Cauchy Goursat Theorem
- Cauchy's integral formula
- Cauchy's integral formula for derivatives

INTRODUCTION

Curve :

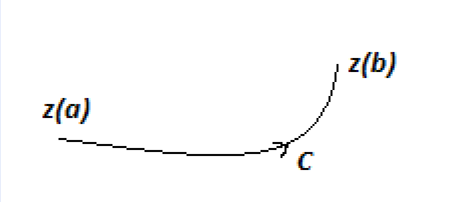
- Range of a continous complex valued function z(t) defined on the interval [a,b]

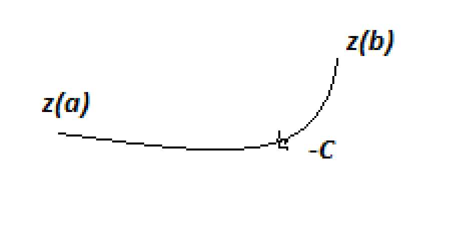
$$z(t) = (x(t), y(t)) = x(t) + iy(t), a \le t \le b$$

- Specify a curve as

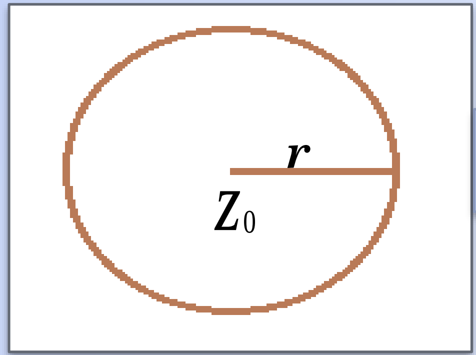
$$C : z(t) = x(t) + iy(t), a \le t \le b$$

$$-C : z(t) = x(-t) + iy(-t), -b \le t \le -a$$

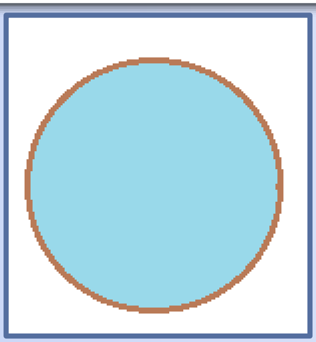




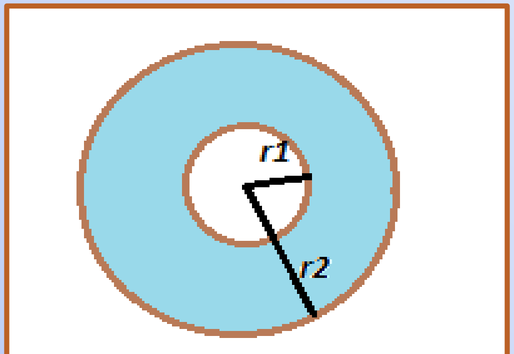
- $z(a) = z(b)$ → closed curve
- $x(t)$ and $y(t)$ both differentiable → Smooth curve
- $z(a) = z(b)$ only point of intersection → Simple closed curve



$$|z - z_0| = r$$



$$D_\in(z_0) = \{z: |z - z_0| < \in\} = open\ circular\ disk$$



$$\{z: r_2 < |z - z_0| < r_1\}$$
$$= circular\ ring\ (\,open\ annulus\,)$$

1. simple curve

2. simple closed curve

3. neither simple nor closed

4. closed but not simple

5. smooth curve

6. not smooth curve

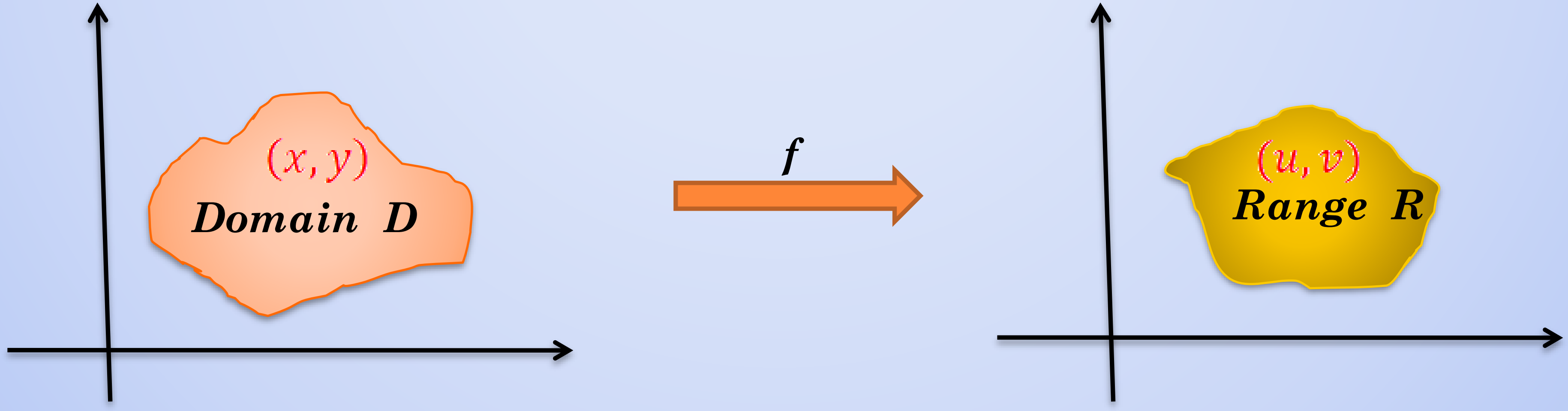
Complex valued function :

$f: D \to \mathbb{C} \quad , where \ D \subset \mathbb{C}$

$f(z) = (u, v) = u + iv$

$\boxed{u : \mathbb{R}^2 \to \mathbb{R} \ , v : \mathbb{R}^2 \to \mathbb{R}}$

$Eg: f(z) = z^2 + 2z \ , e^z$



Complex differentiability :

- U be an open subset of $\mathbb{C}$ *and* $z_0 \in U$
- f is said to be complex differentiable at z_0
  if there exists a complex number L such that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L = f'(z_0)$$

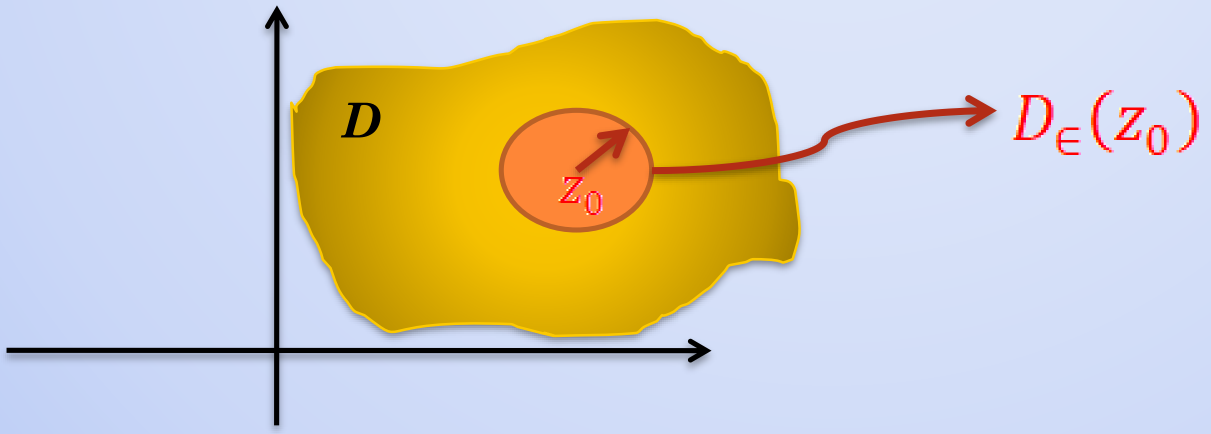
- There are complex functions which are continous everywhere but differentiable nowhere.

Analytic / Holomorphic function :

A function f(z) is said to be analytic at $z = z_0$ *iff there exists some* $\epsilon > 0$ *such that* $f'(z)$ *exists for all* $z \in D_{\epsilon}(z_0)$

i.e. f(z) is differentiable at all points of this $\epsilon -$ *neighbourhood*

- Examples : all polynomial , exponential functions
- **Singular points** : Points of non analyticity of a function



Integral of a complex valued function of a real variable :

- $f(t) = u(t) + iv(t)$, where u and v are real valued functions of real variable t for $a \leq t \leq b$
- Define :

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

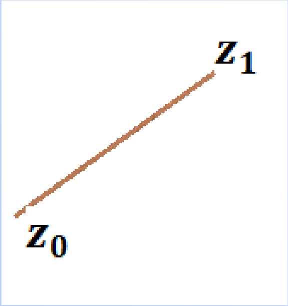
- Complex integrals of this type have properties similar to those of real integrals.

Contour :

- a curve constructed by joining finitely many smooth curves end to end

$$C = C_1 + C_2 + C_3 + \dots + C_n$$

- Equation of a straight line joining $z_0 = (x_0, y_0)$ and $z_1 = (x_1, y_1)$



$$\boxed{z(t) = z_0 + t(z_1 - z_0)}$$

Real Analysis

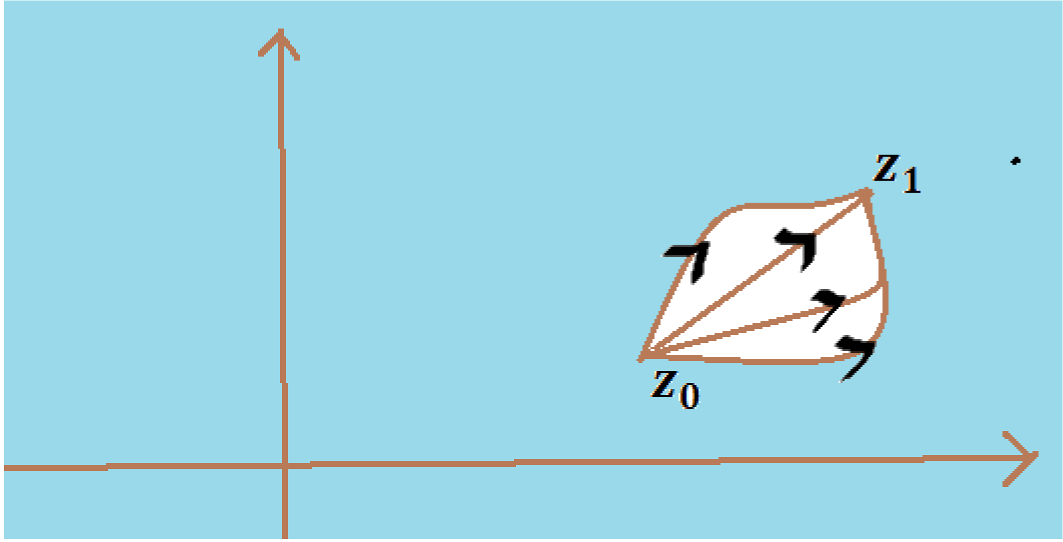
$$\int_a^b f(x)dx$$

- can reach from a to b in only one direction

Complex Analysis

$$\int_{z_0}^{z_1} f(z)dz$$

- can reach from z_0 to z_1 in infinitely many directions



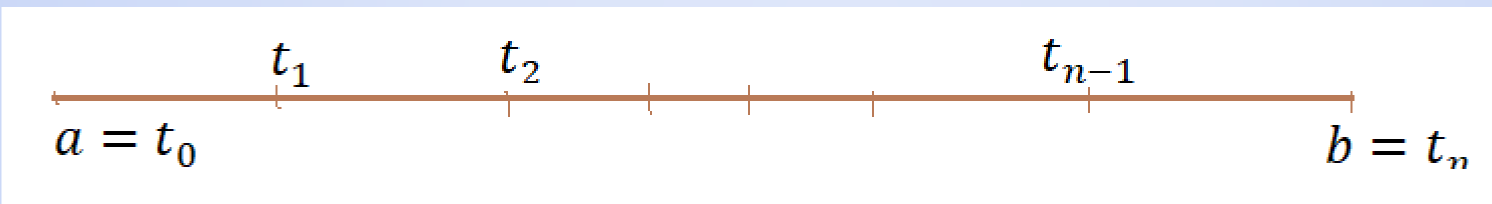
Complex Line Integral :

- C be smooth curve given by

$$z(t) = x(t) + iy(t), \quad a \le t \le b$$

- *f(z)* be continous at each point of C
- Partition the interval $[a, b]$ by points

$$a = t_0, t_1, t_2 \dots \dots t_{n-1}, t_n = b \quad where \ \ t_0 < t_1 < t_2 \dots \dots t_{n-1} < t_n$$



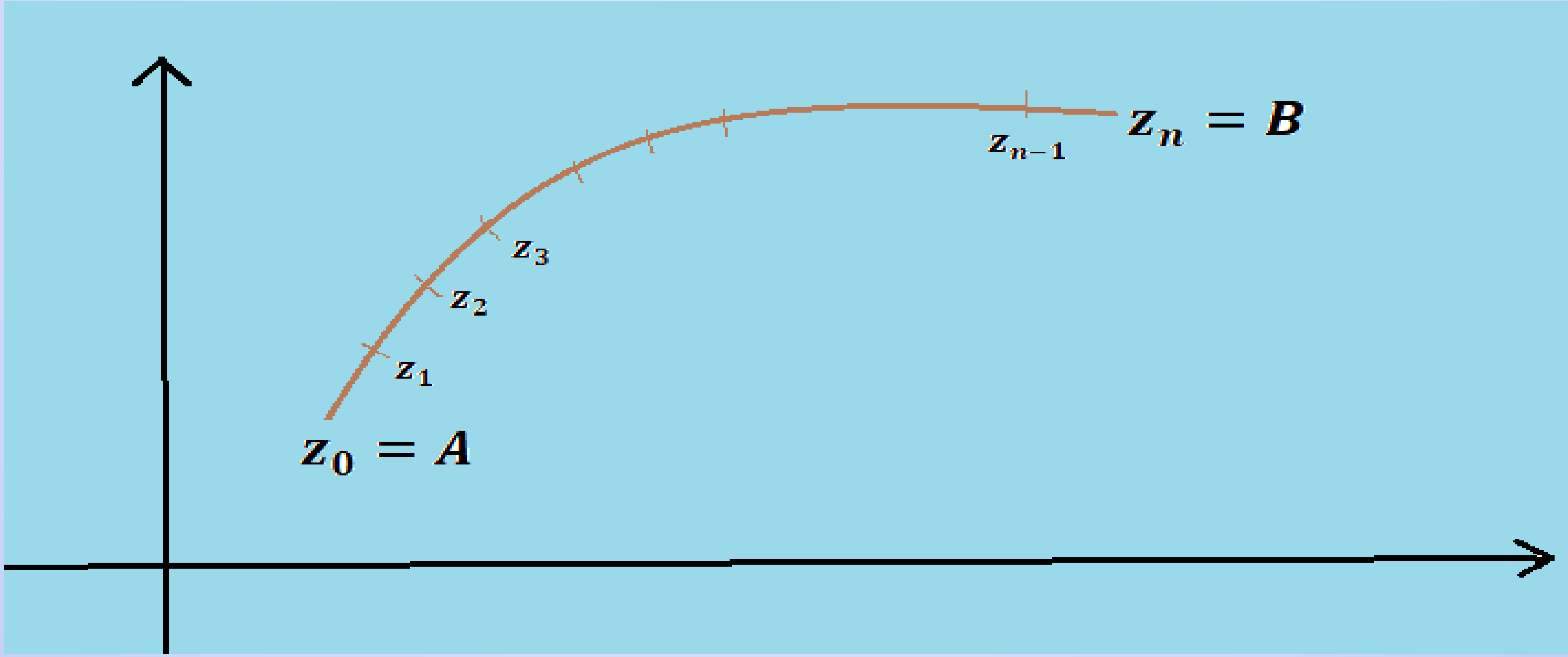
- To this subdivision, there corresponds a sub division of C by points

$$z_0, z_1, z_2, \dots \dots, z_n, \qquad where \ z_j = z(t_j)$$

- Between each pair of partition points z_k *and* z_{k-1} select a point c_k on C, make the Riemann sum

$$S_n = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1})$$

$$= \sum_{k=1}^{n} f(c_k)\Delta z_k \quad , \quad \Delta z_k = z_k - z_{k-1}$$



Defination : Let C be a contour. Then,

$$\int_C f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta z_k$$

, provided the limit exists.

calculation

????

$$= \int_a^b f\left(z(t)\right)z'(t)dt$$

$$\int_C f(z)dz \longrightarrow \textit{Contour integral of } f \textit{ around } C$$

Properties of contour Integration :

1. $\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$

2. $\int_C (a + ib)f(z)dz = (a + ib)\int_C f(z)dz$

3. $\int_{-C} f(z)dz = -\int_C f(z)dz$

4. $\int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

Q1. Evaluate $\int_0^{1+i} (x - y + ix^2)\,dz$

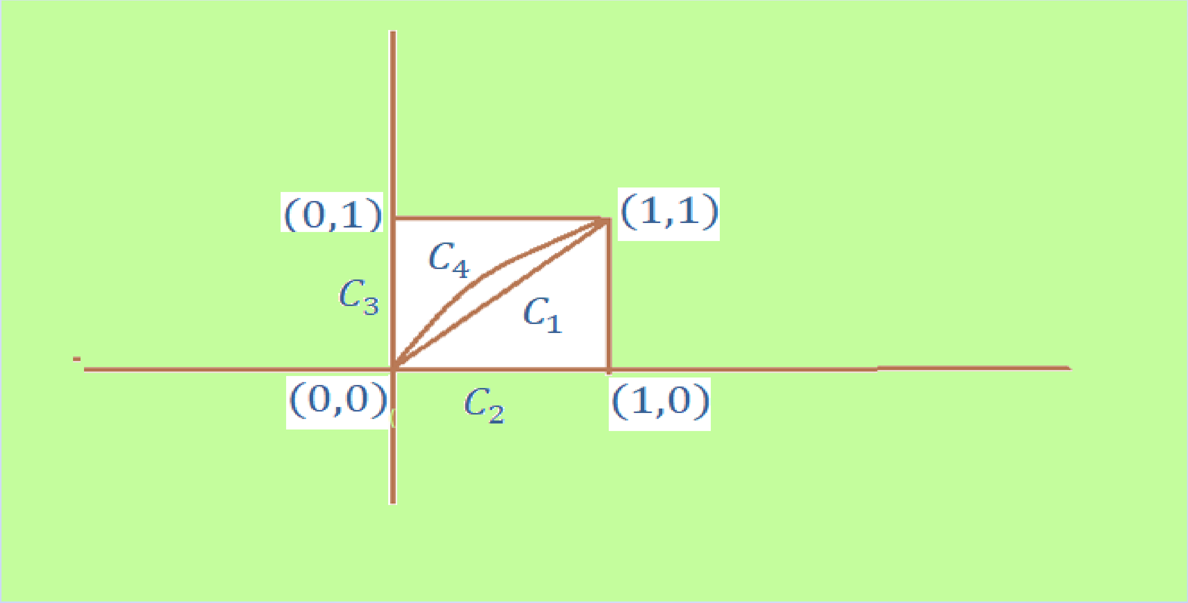
(i) along the straight line from (0,0) to (1,1)

(ii) over the path along the lines $y = 0$ and $x = 1$

(iii) over the path along the lines $x = 0$ and $y = 1$

(iv) along the path $y^2 = x$

Solution :



Use this formula

$$\int_C f(z)dz = \int_a^b f\big(z(t)\big)z'(t)dt$$

(i) $$\int_{C_1} f(z)dz = \frac{1}{3}(i-1)$$

(ii) $$\int_{C_2} f(z)dz = \frac{-1}{2} + \frac{5}{6}i$$

(iii) $$\int_{C_3} f(z)dz = \frac{-1}{2} - \frac{i}{6}$$

(iv) $$\int_{C_4} f(z)dz = \frac{-11}{30} + \frac{i}{6}$$

Q2. Evaluate : $\int_C \frac{2z+3}{z} dz$, where C is

(i) upper half of the circle $|z| = 2$ in the clockwise direction

(ii) lower half of the circle $|z| = 2$ in the anti-clockwise direction, and

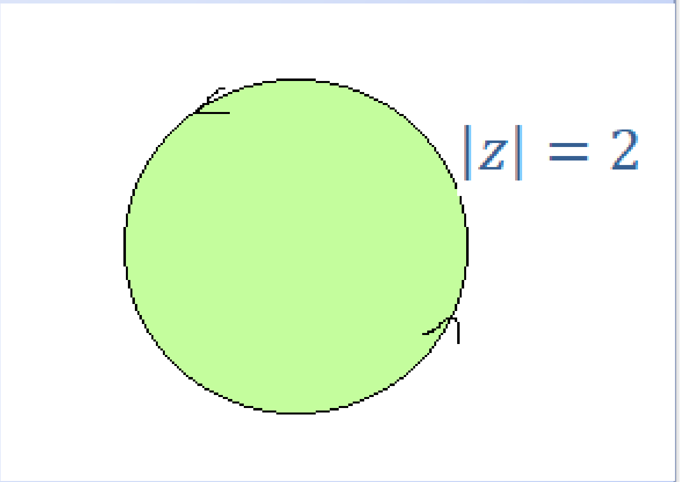
(iii) the circle $|z| = 2$ in the anti-clockwise direction

Solution :

(i) Put $z = 2e^{it}, \quad \pi \le t \le 0$

$$dz = 2ie^{it} dt$$

$$f(z(t)) = \frac{4e^{it} + 3}{2e^{it}}$$

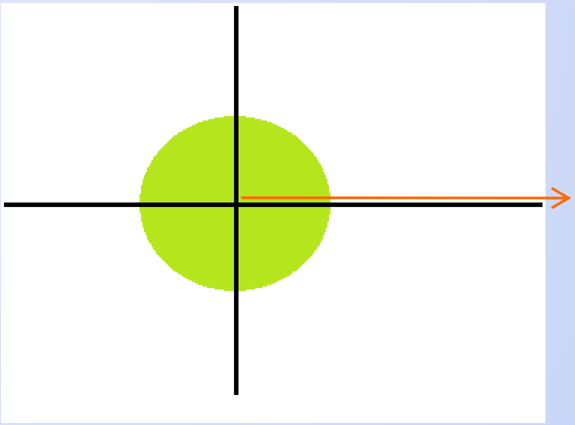


$$\int_C f(z)dz = \int_{\pi}^{0} \frac{4e^{it} + 3}{2e^{it}} 2ie^{it} dt$$

$$= 8 - 3\pi i$$

(ii) $C_2{:}\ \ z = 2e^{it}dt, \quad \pi \le t \le 2\pi$

$$\int_C f(z)dz = \int_{\pi}^{2\pi} \frac{4e^{it} + 3}{2e^{it}} 2ie^{it} dt$$

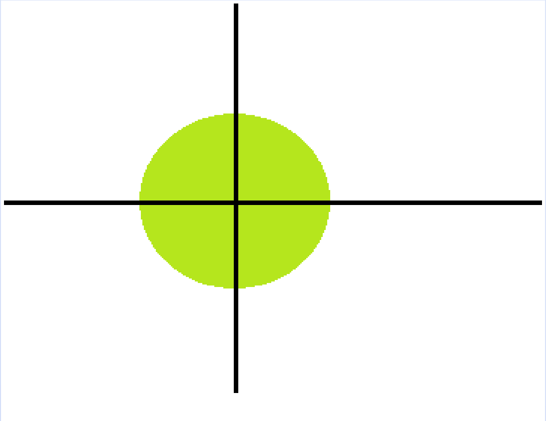


$$= 8 + 3\pi i$$

(iii)

$$C_3: \quad z = 2e^{it}, \quad 0 \le t \le 2\pi$$

$$\int_C f(z)dz = \int_0^{2\pi} \frac{4e^{it} + 3}{2e^{it}} 2ie^{it} dt \quad = 6\pi i$$



Q3. Show that

$$\int_C (z-z_0)^n dz = \begin{cases} 2\pi i, \text{if } n=-1 \\ 0, \quad \text{if } n \neq -1 \end{cases}$$

where C is a circular path with centre z_0 and radius , $r>0$, traversed in the anti-clockwise direction.

Special case

$$\int_C \frac{dz}{z-z_0} = 2\pi i, \qquad where\ C : |z-z_0| = r, r>0$$

result is independent of r

Domain : *open + connected set*

Domain

Simply Connected Domain

$\Downarrow$

"having no holes"

eg. interior of a circle, ellipse

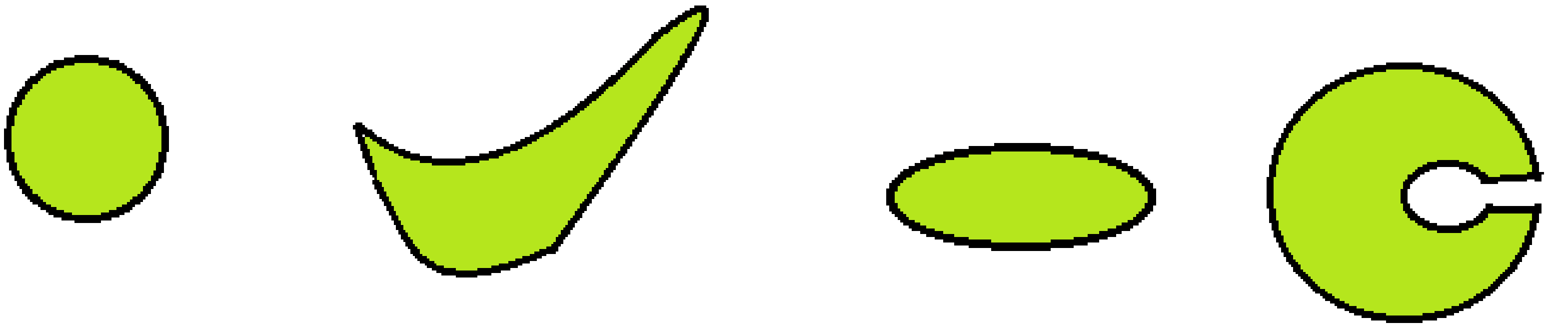
Multiple Connected Domain

$\Downarrow$

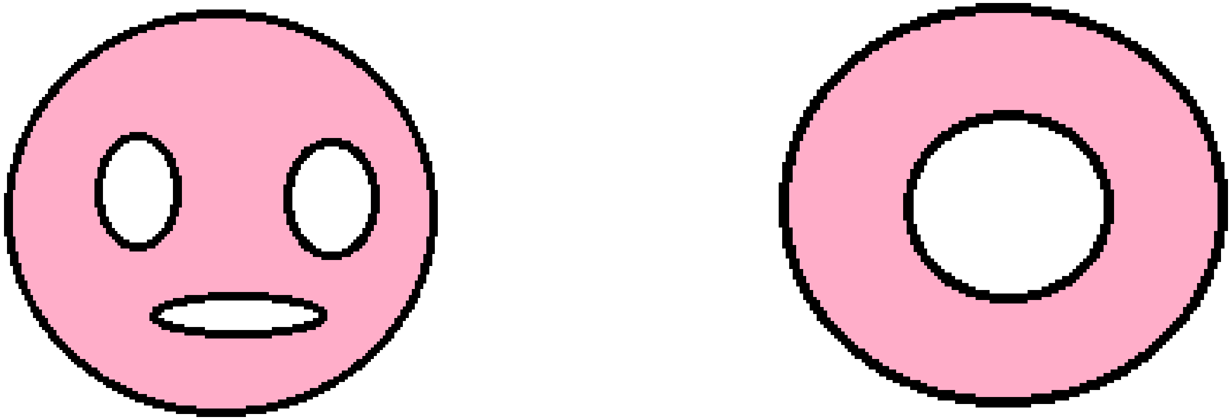
"domain that is not simply connected"

eg. annulus, circular ring

Simply Connected Domain



Multiple Connected Domain



Cauchy - Goursat Theorem

- *Augustin Cauchy first proposed in 1825*
- *Edward Goursat gave a revised proof in 1883*

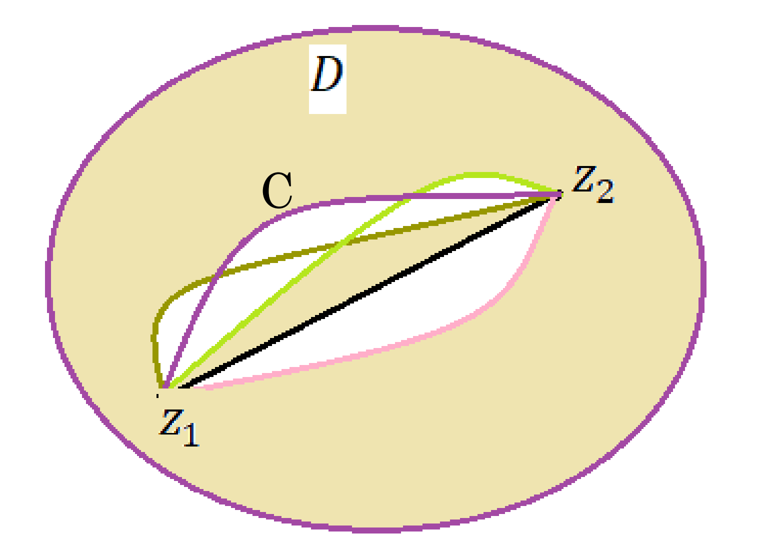
Statement : *Let f(z) be analytic in a simply connected domain D. If C is a simple closed contour that lies in D, then*

$$\int_C f(z)dz = 0$$

**Cauchy proved this theorem with the additional requirement that f' be continous.*

Corollary :

1. ***Independence of path***

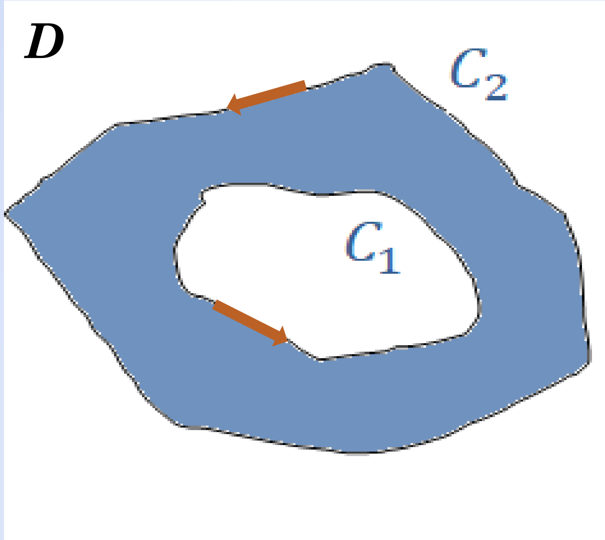


C be any contour joining z_1 and z_2

$$\int_C f(z)dz \quad \text{is independent of curve } C$$

depends only on z_1 and z_2

2. Deformation of contour

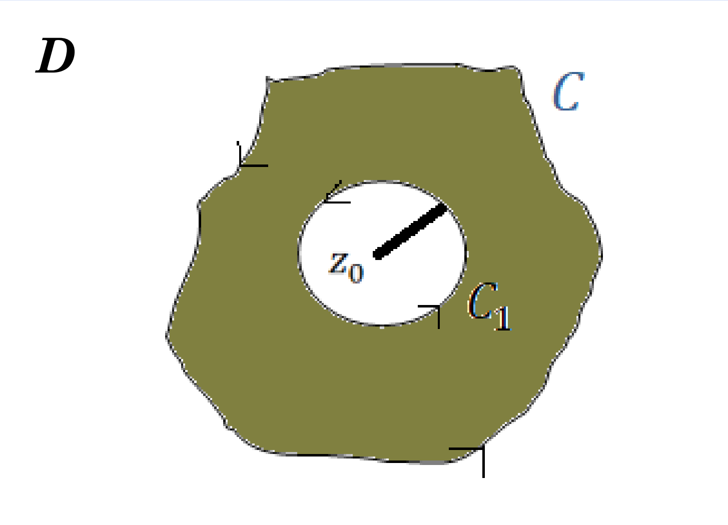


$$f(z) \ is \ analytic \ in \ D$$

$$C_1 \ and \ C_2 \ simple \ closed \ +vely \ oriented \ contour$$

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

conclusion



$$\int_C f(z)dz = \int_{C_1} f(z)dz = 2\pi i$$

C is a simple closed $+ve$ contour

z_0 lies interior to C

Q4. Evaluate : $\int_C \frac{2z}{z^2+2}dz$ where

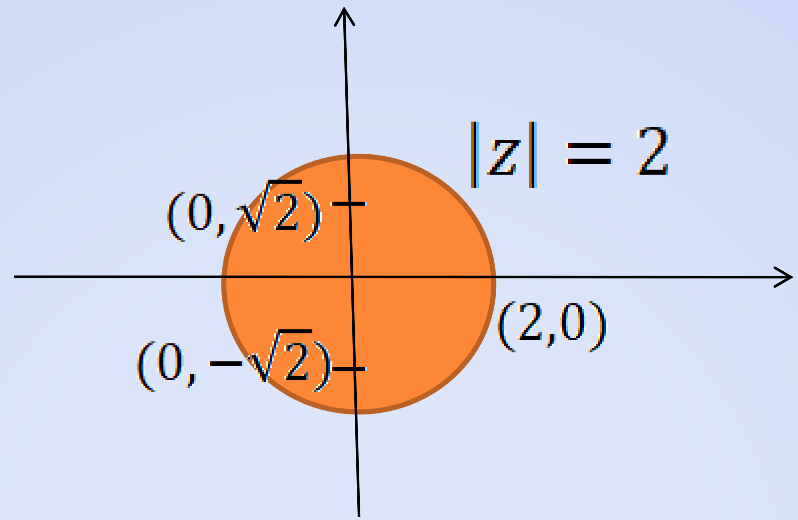
(i) C is the circle $|z| = 2$

(ii) C is the circle $|z - i| = 1$

Solution : By partial fractions

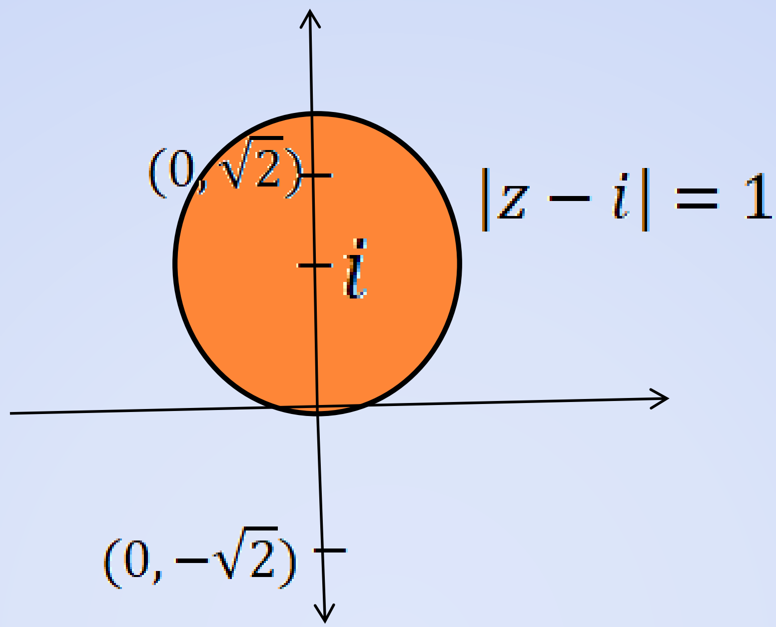
$$\int_C \frac{2z}{z^2+2}dz = \int_C \frac{dz}{z+i\sqrt{2}} + \int_C \frac{dz}{z-i\sqrt{2}}$$

(i) The points $z = i\sqrt{2}, -i\sqrt{2}$ lies interior to C.



By Cauchy Goursat theorem ,

$$\int_C \frac{2z}{z^2+2}\,dz = 2\pi i + 2\pi i = 4\pi i$$



The point $z = i\sqrt{2}$ lies inside C

The point $z = -i\sqrt{2}$ lies outside C

So, the function $\dfrac{1}{z+i\sqrt{2}}$ is analytic inside C, $\displaystyle\int_C \frac{dz}{z+i\sqrt{2}} = 0$

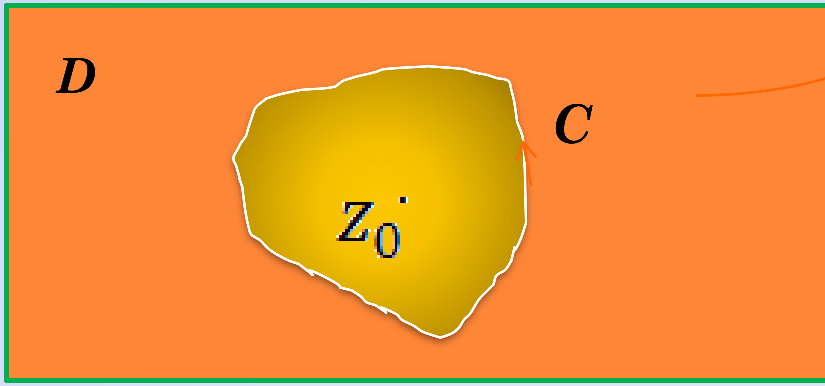
Thus, $\displaystyle\int_C \frac{2z}{z^2+2}\,dz = 0 + 2\pi i = 2\pi i$

Cauchy Integral Formula

- shows that the value of an analytic function f can be represented by a certain contour integral

Statement : *Let f be analytic in the simply connected domain D and let C be a simple closed positively oriented contour that lies in D. If* $\mathbf{z_0}$ *is a point that lies interior to C, then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz$$



f is analytic in D

$$\int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i \times f(z_0)$$

Cauchy Integral Formula for derivatives :

$$f^{n}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

f^{n} denotes nth derivative of f

$$f'(z_0) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{2}}\,dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{3}}\,dz$$ ***and so on***

Consequences :

$\Rightarrow$ $Analytic \;\; functions \; are \; infintely \; differentiable$

$$f(z) = u + iv$$

- If f is analytic in a domain D, then all its derivatives $f', f'', f''', \ldots, f^n$ exist and are analytic in D.
- If f is analytic in a domain D, then all partial derivatives of u and v exist and are continous in D.

Q4. Evaluate : $\oint_C \frac{e^{-z}}{z+1}\,dz$ where C is

(i) $|z| = 2$ (ii) $|z| = \frac{1}{2}$

Solution :

$$\oint_C \frac{e^{-z}}{z+1}\,dz = \oint_C \frac{f(z)}{z-(-1)}\,dz \text{ ,where } f(z) = e^{-z}$$

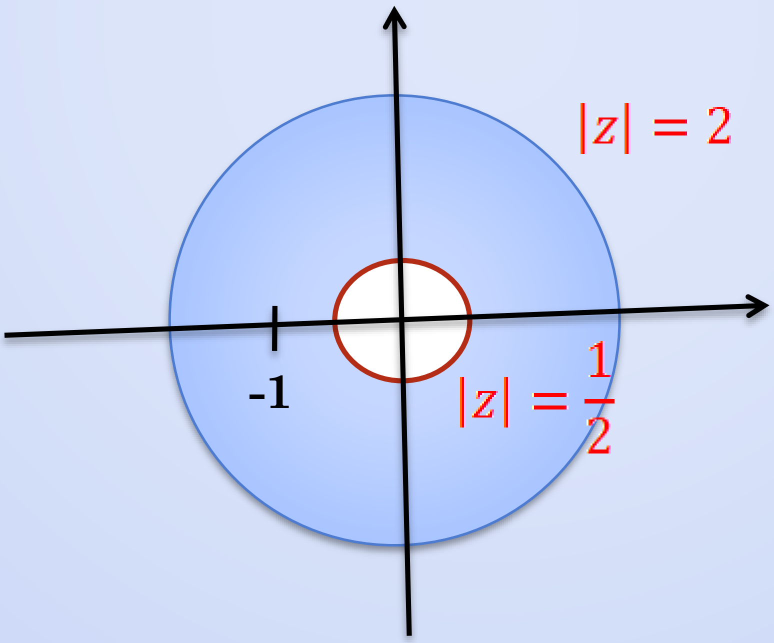
(i) *f(z)* is analytic on and inside $|z| = 2$, and $z_0 = -1$ lies inside $|z| = 2$

By cauchy integral formula,

$$\oint_C \frac{f(z)}{z-(-1)}\,dz = 2\pi i \times f(-1) = 2\pi i e$$

(ii) $\dfrac{e^{-z}}{z+1}$ is analytic on and inside $|z| = \dfrac{1}{2}$

By Cauchy Goursat theorem, $\oint_C \dfrac{e^{-z}}{z+1}\, dz = 0$



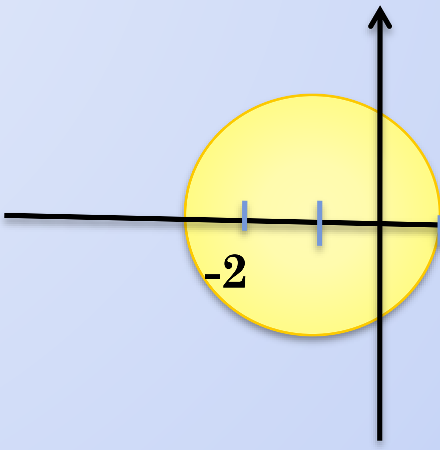
Q5. Evaluate : $\int_C \frac{dz}{(z-1)^2}$ where $C: |z| = 2$

Solution : $\int_C \frac{dz}{(z-1)^2} = 2\pi i f'(1) = 0$ | $f(z) = 1, f'(1) = 0$

Q6. Evaluate : $\int_C \frac{z^2}{4-z^2} dz$ $\quad where\ C: |z+1| = 2$

Solution :

$$\int_C \frac{z^2}{4-z^2} dz = \int_C \frac{z^2}{(2-z)(2+z)} dz$$



$z = 2\ \ lies\ \ outside\ \ C,$

$z = -2\ \ lies\ \ inside\ \ C$

$$\int_C \frac{\frac{z^2}{2-z}}{2+z}\,dz = 2\pi i f(-2) \quad \textit{where } f(z) = \frac{z^2}{2-z}$$

$$= 2\pi i$$

Q7. Evaluate : $\dfrac{1}{2\pi}\displaystyle\int_C \frac{z^2-4}{z^2+4}\,dz \quad \textit{where } C: |z-i| = 2$

Solution : $\dfrac{1}{2\pi}\displaystyle\int_C \frac{z^2-4}{z^2+4}\,dz = \frac{1}{2\pi}\int_C \frac{z^2-4}{(z+2i)(z-2i)}\,dz$

$z = 2i$ *lies inside* C

$z = -2i$ *lies outside* C,

$$= \frac{1}{2\pi}\int_C \frac{\frac{z^2-4}{z+2i}}{z-2i}\,dz$$

$$= \frac{1}{2\pi} \times 2\pi i \times f(2i) \quad where\ f(z) = \frac{z^2 - 4}{z + 2i}$$

$$= -2$$

Q8. Let $f(z) = \sum_{n=0}^{15} z^n$, $z \in \mathbb{C}$. If $C: |z - i| = 2$, then find

$$\int_C \frac{f(z)}{(z-i)^{15}}\, dz$$

Solution :

f(z) being a polynomial of degree 15 is analytical everywhere.

By Cauchy integral formula for derivatives ,

$$\int_C \frac{f(z)}{(z-i)^{15}}\,dz = \frac{2\pi i}{14!} \times f^{14}(i)$$

Now,

$$f(z) = \sum_{n=0}^{15} z^n \ , z \in C$$

$$f^{14}(z) = 14!\,(15z+1)$$

$$\therefore \int_C \frac{f(z)}{(z-i)^{15}}\,dz = \frac{2\pi i}{14!} \times 14! \times (15i+1)$$

$$= 2\pi i(1+15i)$$

Q9. Let $I_r = \int_{C_r} \frac{dz}{z(z-1)(z-2)}$, where $C_r = \{z \in \mathbb{C} : |z| = r\}$

Then,

$(a)\ I_r = 2\pi i, \quad if\ r \in (2,3)$

$(b)\ I_r = \frac{1}{2}, \quad if\ r \in (0,1)$

$(c)\ I_r = -2\pi i, \quad if\ r \in (1,2)$

$(d)\ I_r = 0, \quad if\ r > 3$

Choose the correct one.

Solution :

Partial fractions

$$I_r = \int_{C_r} \frac{dz}{z(z-1)(z-2)} = \frac{1}{2}\int_{C_r} \frac{dz}{z} - \int_{C_r} \frac{dz}{z-1} + \frac{1}{2}\int_{C_r} \frac{dz}{z-2} \qquad .....*$$

(a) $If\ r \in (2,3)$, $z = 0, 1, 2\ all\ lie\ inside\ C_r$

$$*\ gives\ I_r = \frac{1}{2} \times 2\pi i - 2\pi i + \frac{1}{2} \times 2\pi i = -\pi i$$

so, option (a) is incorrect.

(b) $If\ r \in (0,1)$ $z = 1\ and\ 2\ lies\ outside\ C_r$

$z = 0\ lies\ inside\ C_r$

$$*\ gives\ I_r = \frac{1}{2} \times 2\pi i - 0 + \frac{1}{2} \times 0 = \pi i$$

so, option (b) is incorrect.

(c) $If\ r \in (1,2)$

$z = 0\ \ and\ \ 1\ \ all\ \ lie\ \ inside\ \ C_r \qquad z = 2\ \ lies\ outside\ \ C_r$

$$*\ \ gives\ \ I_r = \frac{1}{2} \times 2\pi i - 2\pi i + \frac{1}{2} \times 0 = -\pi i$$

so, option (c) is incorrect.

(d) $If\ r > 3$

$z = 0, 1, 2\ \ all\ \ lie\ \ inside\ \ C_r$

$$*\ \ gives\ \ I_r = \frac{1}{2} \times 2\pi i - 2\pi i + \frac{1}{2} \times 2\pi i = 0$$

so, option (d) is correct.

Q10. If $\int_C \left(\frac{z+1}{z^2-3z+2}+\frac{a}{z-1}\right)dz = 0, where\ \ C : |z| = \frac{3}{2}$

find the value of a.

Solution: $\int_C \left(\frac{z+1}{z^2-3z+2}+\frac{a}{z-1}\right)dz = 0$

$or, \int_C \left(\frac{z+1}{(z-1)(z-2)}+\frac{a}{z-1}\right)dz = 0$

$z = 1\ lies\ \ inside\ \ C$

$z = 2\ \ lies\ outside\ \ C$

$or, \int_C \frac{\frac{z+1}{z-2}}{z-1}dz + a\int_C \frac{dz}{z-1} = 0 \qquad where\ \ f(z) = \frac{z+1}{z-2}$

$or, 2\pi i \times f(1) + a \times 2\pi i = 0$

$\Downarrow$

$a = 2$

Q11. Evaluate : $\frac{1}{2\pi i}\int_C |1+z+z^2|^2\,dz$, *where* $C: |z| = 1$

Solution :

$$\frac{1}{2\pi i}\int_C |1+z+z^2|^2\,dz$$

$$= \frac{1}{2\pi i}\int_C (1+z+z^2)\left(\overline{1+z+z^2}\right)dz$$

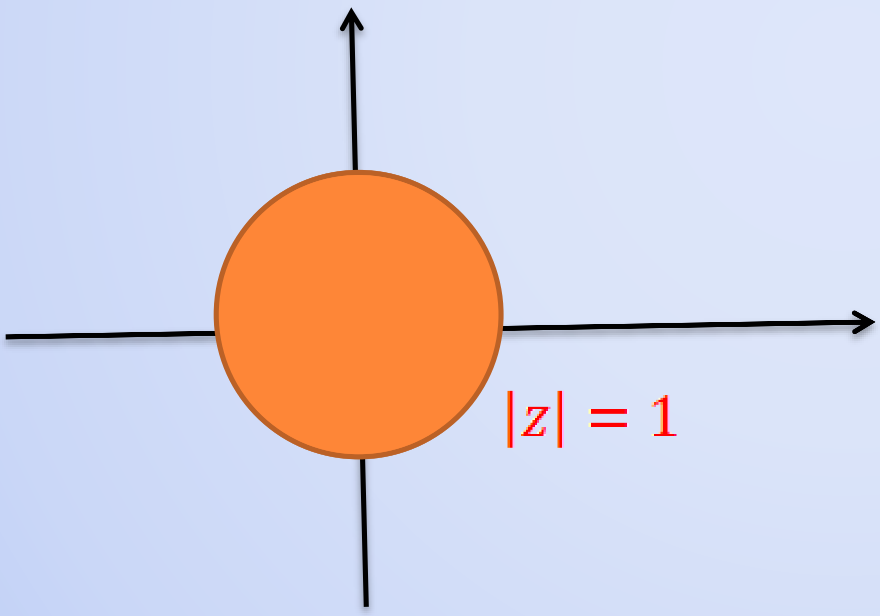
$$= \frac{1}{2\pi i}\int_C (1+z+z^2)\left(1+\bar{z}+\bar{z}^2\right)dz$$

$$= \frac{1}{2\pi i}\left[\int_C (1+z+z^2)\,dz + \int_C (\bar{z}+z\bar{z}+\bar{z}z^2)\,dz + \int_C \left(\bar{z}^2+z\bar{z}^2+z^2\bar{z}^2\right)dz\right]$$

$$= \frac{1}{2\pi i}\left[0 + \int_C \left(\frac{1}{z} + z \times \frac{1}{z} + \frac{1}{z} \times z^2\right) dz + \int_C \left(\frac{1}{z^2} + z \times \frac{1}{z^2} + \frac{1}{z^2} \times z^2\right) dz\right]$$

$$Along \;\; C, \qquad \bar{z} = \frac{1}{z}$$

$$= 2$$

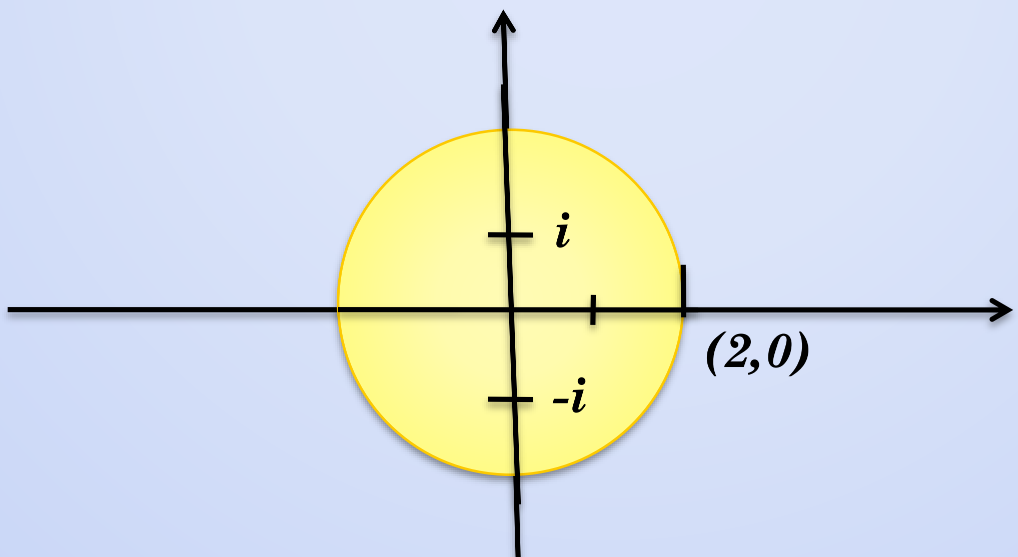


Q12. Evaluate : $\int_C \frac{\cosh(\pi z)}{z(z^2+1)} dz \quad , where \quad C: |z| = 2$

Solution :

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)} dz = \int_C \frac{\cosh(\pi z)}{z} dz - \frac{1}{2}\int_C \frac{\cosh(\pi z)}{z+i} dz \quad - \frac{1}{2}\int_C \frac{\cosh(\pi z)}{z-i} dz$$

The points $z = 0, i, -i$ *all lie inside* $|z| = 2.$



$By\ \ Cauchy\ \ integral\ \ formula,\qquad where\ f(z) = \cosh(\pi z)$

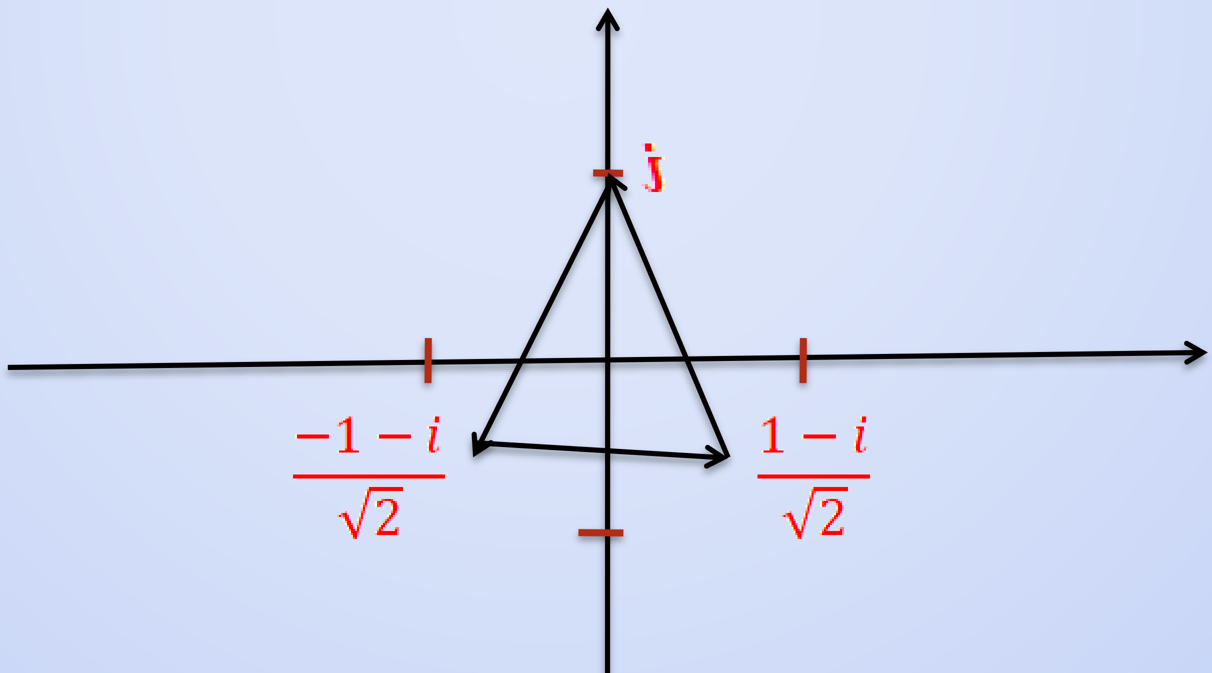
$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)}\,dz \;= 2\pi i \times f(0) - \frac{1}{2} \times 2\pi i \times f(-i) - \frac{1}{2} \times 2\pi i \times f(i)$$

$$= 2\pi i\left[f(0) - \frac{1}{2} \times f(-i) - \frac{1}{2} \times f(i)\right]$$

$$= 4\pi i$$

Q13. $Let \int_C \left[\frac{1}{(z-2)^4} - \frac{(a-2)^2}{z} + 4\right] dz = 4\pi$ where the closed curve C is a triangle having vertices at $i, \left(\frac{-1-i}{\sqrt{2}}\right)$ and $\left(\frac{1-i}{\sqrt{2}}\right)$. Find the values of a.

Solution :



$$\int_C \left[\frac{1}{(z-2)^4} - \frac{(a-2)^2}{z} + 4\right] dz = 4\pi$$

$$or, \int_C \frac{dz}{(z-2)^4} - (a-2)^2 \int_C \frac{dz}{z} + 4\int_C dz = 4\pi$$

$$or, 0 - (a-2)^2 \int_C \frac{dz}{z-0} + 0 = 4\pi \qquad \Big| \; as\ \ z = 2\ \ lies\ \ outside\ \ C$$

$$or, a = 3+i, 1-i$$

Q14. Evaluate : $\int_C \frac{e^{2z}}{(z+1)^4} dz \quad , where\ C: |z| = 2$

Solution : $z = -1\ lies\ inside\ C$

$f(z) = e^{2z}\ is\ analytic\ inside\ and\ on\ |z| = 2$

By Cauchy integral formula,

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \int_C \frac{e^{2z}}{(z-(-1))^4} dz,$$

$$= \frac{2\pi i}{3!} \times f'''(-1)$$

$$= \frac{8\pi i}{3e^2}$$

Q15. Evaluate : $\int_C \frac{e^z}{z(1-z)^3}\,dz$, where C is

$(i)\ \ |z| = \frac{1}{2}$

$(ii)\ \ |z-1| = \frac{1}{2}$

$(iii)\ \ |z| = 2$

Solution :

$$\int_C \frac{e^z}{z(1-z)^3}\,dz = \int_C e^z\left(\frac{1}{z} - \frac{1}{z-1} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3}\right)dz$$

$$= \int_C \frac{e^z}{z}\,dz - \int_C \frac{e^z}{z-1}\,dz + \int_C \frac{e^z}{(z-1)^2}\,dz - \int_C \frac{e^z}{(z-1)^3}\,dz$$

.....*

(1) $z=0$ *lies inside* $|z|=\dfrac{1}{2}$, $z=1$ *lies outside* $|z|=\dfrac{1}{2}$

so, (∗) gives

$$\int_C \frac{e^z}{z(1-z)^3}\,dz = 2\pi i\times f(0)+0+0+0 \qquad , f(z)=e^z$$

$$= 2\pi i$$

(2) $z=0$ *lies outside* $|z-1|=\dfrac{1}{2}$ $\quad z=1$ *lies inside* $|z-1|=\dfrac{1}{2}$

so, (∗) gives

$$\int_C \frac{e^z}{z(1-z)^3}\,dz = 0+-2\pi i\times f(1)+2\pi i\times f'(1)-\frac{2\pi i}{2!}\times f''(1)$$

$$= -\pi i e$$

(3) $z = 0$ *and* 1 *lies inside* $|z| = 2$

so, $(*)$ *gives*

$$\int_C \frac{e^z}{z(1-z)^3}dz = \int_C \frac{e^z}{z}dz - \int_C \frac{e^z}{z-1}dz + \int_C \frac{e^z}{(z-1)^2}dz - \int_C \frac{e^z}{(z-1)^3}dz$$

$$= 2\pi i \times f(0) - 2\pi i \times f(1) + 2\pi i \times f'(1) - \frac{2\pi i}{2!} \times f''(1)$$

$$= 2\pi i - 2\pi i e + 2\pi i e - \pi i e$$

$$= \pi i(2 - e)$$

Thank You