Power Series : An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

is called a power series, with z_0 as the point of expansion. The constants a_n are called coefficients of the power series.

Example : The geometric series $\sum_{n=0}^{\infty} z^n$ converges if $|z| < 1$

We ask the following questions.

1. For what values of z does the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges ?

2. What properties can be attributed to $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ at points where the series converge ?

3. Under what condition may a function *f(z)* be represented by a power series in some neighbourhood of a point ?

Theorem 1 : Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series. Then, any one of the following holds :

(i) either it is absolutely convergent for all $z \in \mathbb{C}$

(ii) or, there is a unique non negative real number R such that

(a) $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent for all $z \in \mathbb{C}$ with $|z| < R$, and

(b) $\sum_{n=0}^{\infty} a_n z^n$ is divergent for all $z \in \mathbb{C}$ with $|z| > R$

The unique $R > 0$ is called the radius of convergence of the power series. The circle $|z| = R$ is called the circle of convergence of the power series.

Proof : Read yourself

Generalisation : Consider the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

According to theorem 1, the given series is absolutely convergent for all $z \in \mathbb{C}$ with $|z - z_0| < R$, and divergent for all $z \in \mathbb{C}$ with $|z - z_0| > R$.

Uniform Convergence : The sequence $\{S_n(z)\}$ is said to converge uniformly to *f(z)* on the set T if for every $\epsilon > 0$ $\exists$ a positive integer N(depending only on ϵ) such that if $n \geq N$, then $|S_n(z)| < \epsilon$ for all $z \in T$.

Weierstrass M-test : Let the infinite series $\sum_{n=0}^{\infty} u_n(z)$ have the property that for each n, $|u_n(z)| \leq M_n$ for all $z \in T$. If $\sum_{n=0}^{\infty} M_n$ converges, then $\sum_{n=0}^{\infty} u_n(z)$ converges uniformly on T.

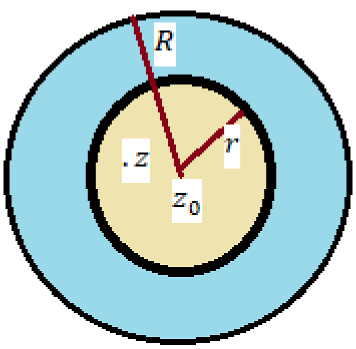
Theorem 2 : A power series is uniformly convergent within its circle of convergence.

or

Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n\,(z - z_0)^n$. Then, for each $r, 0 < r < R$,the series converges uniformly on the closed disk

$\bar{B}(z_0, r) = \{ z: |z - z_0| \leq r\}$

Proof :



Choose $0 < r < R$. Then, there exists z_1 such that

$r < |z_1 - z_0| < R$ and the series $\sum_{n=0}^{\infty} a_n\,(z_1 - z_0)^n$ is convergent.

So, the sequence $\{ a_n(z_1 - z_0)^n \}$ converges to zero as $n \to \infty$ and hence bounded.

So, there exists a constant M > 0 such that $|a_n(z_1 - z_0)^n| \leq M \;\; \forall\, n \geq 0$ ….(1)

Let $z \in \bar{B}(z_0, r)$. Then, $|z - z_0| \leq r < |z_1 - z_0|$

$$\text{or, } \frac{|z-z_0|}{|z_1-z_0|} = \rho\,(\,say\,) < 1.$$

But, $|a_n(z-z_0)^n| = |a_n||z-z_0|^n = |a_n||z_1-z_0|^n \frac{|z-z_0|^n}{|z_1-z_0|^n}$

$$= |a_n(z_1-z_0)^n|\left|\frac{z-z_0}{z_1-z_0}\right|^n \le M\rho^n \;, n \ge 0$$

Now, the series $\sum_{n=0}^{\infty} M\rho^n$ is a geometric series with common ratio $\rho < 1$ and hence convergent.

So, by Weierstrass M-test , the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly in $\bar{B}(z_0, r)$.

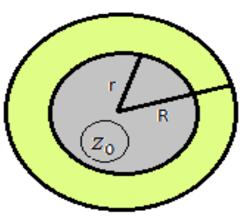
The next theorem shows that every function defined by its power series is analytic inside its circle of convergence.

Theorem 3 : Let $R > 0$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < R$ .Then, $f(z)$ is analytic for $|z| < R$ with $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$.

Proof: The series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is absolutely convergent for $|z| < R$. (Why ?..)

Let $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$, $|z| < R$.

We show that $f'(z_0) = g(z_0)$ for $|z_0| < R$. i.e $\lim_{z \to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0} - g(z_0)\right) = 0$.



Choose r such that $|z_0| < r < R$.

As $z \to z_0$, we restrict z so that $|z| < r$.

Let $\epsilon > 0$. As, $\sum_{n=1}^{\infty} n a_n r^{n-1}$ converges absolutely, $\exists$ a positive integer N such that $\sum_{n=N+1}^{\infty} |n a_n r^{n-1}| < \frac{\epsilon}{4}$ .....(1)

Keeping N fixed and $z \ne z_0$, we have

$$\frac{f(z)-f(z_0)}{z-z_0} - \sum_{n=1}^{\infty} n a_n {z_0}^{n-1} = \sum_{n=1}^{\infty} a_n \frac{(z^n - {z_0}^n)}{z-z_0} - \sum_{n=1}^{\infty} n a_n {z_0}^{n-1}$$

$$= \sum_{n=1}^{\infty} a_n\left(z^{n-1} + z^{n-2}z_0 + \cdots + {z_0}^{n-1}\right) - \sum_{n=1}^{\infty} n a_n {z_0}^{n-1}$$

$$= \sum_{n=1}^{\infty} a_n\left(z^{n-1} + z^{n-2}z_0 + \cdots + {z_0}^{n-1} - n{z_0}^{n-1}\right)$$

$$= \sum_{n=1}^{N} a_n\left(z^{n-1} + z^{n-2}z_0 + \cdots + {z_0}^{n-1} - n{z_0}^{n-1}\right) + \sum_{n=N+1}^{\infty} a_n\left(z^{n-1} + z^{n-2}z_0 + \cdots + {z_0}^{n-1} - n{z_0}^{n-1}\right)$$

or, $\frac{f(z)- f(z_0)}{z-z_0}$ - $\sum_{n=1}^{\infty} n a_n {z_0}^{n-1} = S_1 + S_2$

or, $\left|\frac{f(z)- f(z_0)}{z-z_0} - \sum_{n=1}^{\infty} n a_n {z_0}^{n-1}\right| < |S_1| + |S_2|$ ......(2)

Now,

$$\left|a_n\left(z^{n-1} + z^{n-2}z_0 + \cdots + {z_0}^{n-1} - n{z_0}^{n-1}\right)\right| \le |a_n|\left(\left|z^{n-1} + z^{n-2}z_0 + \cdots + {z_0}^{n-1}\right| + n|z_0|^{n-1}\right)$$

$$\le |a_n|\left(|z|^{n-1} + |z|^{n-2}|z_0| + \cdots + |z_0|^{n-1} + n|z_0|^{n-1}\right)$$

$$\le |a_n|\left(r^{n-1} + r^{n-2}r + \cdots + r^{n-1} - nr^{n-1}\right)$$

$$= |a_n|(nr^{n-1} + nr^{n-1})$$

$$= 2|a_n|\, nr^{n-1}$$

$\therefore\ |S_2| \le \sum_{n=N+1}^{\infty} 2\,|a_n|\, nr^{n-1} < \frac{\epsilon}{2}$ [From (1)]

Also, $S_1 = \sum_{n=1}^{N} a_n\left(z^{n-1} + z^{n-2}z_0 + \cdots + {z_0}^{n-1} - n{z_0}^{n-1}\right)$ is a polynomial in z.

$$\therefore \lim_{z\to z_0} S_1 = \lim_{z\to z_0}\left(\sum_{n=1}^{N} a_n\left(z^{n-1} + z^{n-2}z_0 + \cdots + {z_0}^{n-1} - n{z_0}^{n-1}\right)\right)$$

$$= \sum_{n=1}^{N} a_n\left({z_0}^{n-1} + {z_0}^{n-2}z_0 + \cdots + {z_0}^{n-1} - n{z_0}^{n-1}\right) = 0$$

So, for a given $\epsilon > 0$, $\exists\, \delta > 0$ such that

$|S_1 - 0| < \frac{\epsilon}{2}$ whenever $0 < |z - z_0| < \delta$.

Thus, for $|z| < r$ and $|z - z_0| < \delta$, we have

$\left|\frac{f(z)- f(z_0)}{z-z_0} - g(z_0)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ [From (2)]

$\Rightarrow \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = g(z_0)$

or, $f'(z_0) = g(z_0)$.

Thus, *f(z)* is analytic inside $|z| < R$ *(Proved)*

Corollary : Theorem 3 can be repeatedly applied to obtain,

$$f^k(z) = \sum_{n=k}^{\infty} n(n-1)(n-2) \dots (n-k+1) a_n z^{n-k}$$

$$= k!\, a_k + \frac{(k+1)!}{1!} a_{k+1} z + \frac{(k+2)!}{2!} a_{k+2}\, z^2 + \cdots, \quad |z| < R$$

Setting $z = 0$, we observe the coeffecients a_k are associated with the sum function through the following expressions,

$$f^k(0) = k!\, a_k$$

or, $a_k = \frac{f^k(0)}{k!}$.

The representation $f(z) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} z^k$ is called the Maclaurin series representation of $f(z)$.

Taylor Series :

We saw that the complex power series $\sum_{n=0}^{\infty} a_n\,(z - z_0)^n$ is analytic in their region of convergence $|z - z_0| < R$, where R is the radius of convergence.

We now prove the converse part i.e. if $f(z)$ is analytic in the disk $|z - z_0| < R$, then $f(z)$ can be represented by a power series of the form

$f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n$.

Thus, every holomorphic function defined in a domain D possesses a power series expansion in a disk around any point $z_0 \in D$.

Remainder after n terms :

$S = S_N + R_N$,

Where S = sum of the series, $S_N =$ sum of the series after N terms, $R_N =$ remainder after N terms.

$\therefore |S_N - S| = |R_N - 0|$

We observe that a series converges to a number S if and only if the sequence of remainders tends to zero.

Defination of Taylor series : If $f(z)$ is analytic at $z = z_0$, then the series

$$f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \frac{f'''(z_0)}{3!}(z - z_0)^3 + \cdots \quad = \sum_{k=0}^{\infty} \frac{f^k(z_0)}{k!}(z - z_0)^k$$

is called the Taylor series for f centered at z_0. When the center is $z_0 = 0$, the series is called Maclaurin series for f.

Taylor's Theorem : Let f be analytic in a domain D and let $z_0 \in D$, $B(z_0, R)$ be any disk contained in D. Then, the Taylor series for f converges to $f(z)$ for all z in $B(z_0, R)$ i.e.

$$f(z) = \sum_{k=0}^{\infty} \frac{f^k(z_0)}{k!}(z - z_0)^k \quad , \text{for all } z \in B(z_0, R).$$

Moreover, for any r, $0 < r < R$, the convergence is uniform on the closed disk $\bar{B}(z_0, r) = \{ z : |z - z_0| \leq r\}$.

or

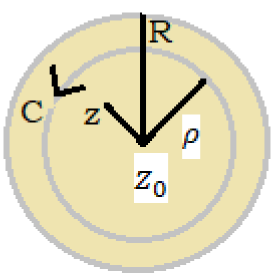
If $f(z)$ is analytic in a circular domain D with center z_0 , then for every z in D, $f(z)$ can be expressed as a power series about z_0.

i.e. $f(z) = \sum_{n=0}^{\infty} a_n \, (z - z_0)^n$, where $a_n = \dfrac{f^n(z_0)}{n!}$

Proof: Let $z \in B(z_0, R)$ and let r denotes the distance between z_0 and z i.e. $|z - z_0| = r$.

Clearly, $0 \leq r < R$.

Choose ρ such that $0 \leq r < \rho < R$ and let C be a positively oriented circle centered at z_0 and radius ρ.



By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} d\,\xi \qquad \ldots (1)$$

Now,

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0}\left[\frac{1}{1 - \frac{z - z_0}{\xi - z_0}}\right]$$

$$= \frac{1}{\xi - z_0}\left[1 - \frac{z - z_0}{\xi - z_0}\right]^{-1}$$

Let $\frac{z-z_0}{\xi-z_0} = w$. Then, $|w| = \frac{|z-z_0|}{|\xi-z_0|} = \frac{r}{\rho} < 1$

So, $\frac{1}{1-\frac{z-z_0}{\xi-z_0}} = \frac{1}{1-w} = 1 + w + w^2 + w^3 + \cdots + w^{n-1} + \frac{w^n}{1-w}$

$$=1+ \frac{z-z_0}{\xi-z_0} + \frac{(z-z_0)^2}{(\xi-z_0)^2} + \cdots + \frac{(z-z_0)^{n-1}}{(\xi-z_0)^{n-1}} + \frac{(z-z_0)^n}{(\xi-z_0)^{n-1}(\xi-z)}$$

So, (1) gives

$$f(z) = \frac{1}{2\pi i}\int_C \left[\frac{1}{\xi-z_0} + \frac{z-z_0}{(\xi-z_0)^2} + \frac{(z-z_0)^2}{(\xi-z_0)^3} + \cdots + \frac{(z-z_0)^{n-1}}{(\xi-z_0)^n} + \frac{(z-z_0)^n}{(\xi-z_0)^n(\xi-z)}\right] f(\xi)\, d\xi$$

or, $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi-z_0} d\xi + \frac{z-z_0}{2\pi i}\int_C \frac{f(\xi)}{(\xi-z_0)^2} d\xi + \cdots + \frac{(z-z_0)^{n-1}}{2\pi i}\int_C \frac{f(\xi)}{(\xi-z_0)^n} d\xi +$
$\frac{(z-z_0)^n}{2\pi i}\int_C \frac{f(\xi)}{(\xi-z_0)^n(\xi-z)} d\xi$

or, $f(z) = f(z_0) + (z-z_0)\frac{f'(z_0)}{1!} + (z-z_0)^2\frac{f''(z_0)}{2!} + \cdots + (z-z_0)^{n-1}\frac{f^{n-1}(z_0)}{(n-1)!} +$
$R_n(z)$,

where $R_n(z) = \frac{(z-z_0)^n}{2\pi i}\int_C \frac{f(\xi)}{(\xi-z_0)^n(\xi-z)} d\xi$

or, $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_{n-1}(z-z_0)^{n-1} + R_n(z)$,

where $a_k = \frac{f^k(z_0)}{k!}$

We next show that $R_n(z) \to 0$ *as* $n \to \infty$.

From our constructions, $|z-z_0| = r$, $|\xi - z_0| = \rho$

$\therefore$ $|\xi - z| = |(\xi - z_0) - (z - z_0)|$

$\geq |\xi - z_0| - |z - z_0|$

$= \rho - r$

Let $M = \max_{\xi\in C} |f(\xi)|$

$\therefore\ |R_n(z)| = \left|\frac{1}{2\pi i}\int_C \frac{f(\xi)\,(z-z_0)^n}{(\xi-z_0)^n(\xi-z)}\,d\xi\right|$ .....(2)

Now, $\left|\frac{f(\xi)\,(z-z_0)^n}{(\xi-z_0)^n(\xi-z)}\right| = \left|\frac{(z-z_0)^n}{(\xi-z_0)^n}\right|\left|\frac{f(\xi)}{\xi-z}\right| \le \left(\frac{r}{\rho}\right)^n \frac{M}{\rho-r}$

Length of C is $2\pi\rho$.

So, by using ML inequality in (2), we get

$|R_n(z)| \le \frac{1}{2\pi}\left(\frac{r}{\rho}\right)^n \frac{M}{\rho-r}(2\pi\rho) \to 0$ $as\ n \to \infty$, as $\frac{r}{\rho} < 1$.

Thus, the series $\sum_{k=0}^{\infty}\frac{f^k(z_0)}{k!}(z-z_0)^k$ converges to $f(z)$, for all $z \in B(z_0, R)$.

Now, the radius of convergence of the series $\sum_{k=0}^{\infty}\frac{f^k(z_0)}{k!}(z-z_0)^k$ is atleast R, which implies that the power series converges uniformly on every closed disk $\bar{B}(z_0, r)$, where $0 \le r < R$.

Theorem 4 : The function $f(z)$ is analytic at z_0 if and only if it can be expanded in Taylor series at z_0.

Proof: Combine proofs of Theorem 3 and Taylor's theorem.

Taylor series expansion of some well known functions :

1. $f(z) = e^z$

2. $f(z) = Log\ z$

3. $f(z) = z^2 e^{3z}$

4. $f(z) = sinz$

5. $f(z) = cosz$

6. $f(z) = sinhz$

7. $f(z) = coshz$

8. $f(z) = \frac{1}{1-z}$

9. $f(z) = \frac{1}{1+z}$

10. $f(z) = \frac{z^2-1}{(z+2)(z+3)}, \ |z| < 2$

11. $f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)}, |z| < 1$

12. $f(z) = \frac{1}{z}$ $about \ \ z = 1$

13. $f(z) = \frac{2z^3+1}{z^2+z}$ $about \ \ z = 1$

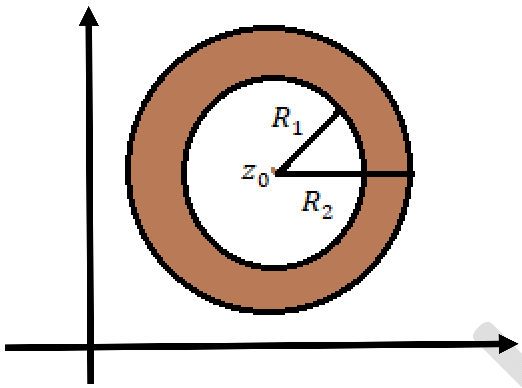
14. $f(z) = \frac{1}{(z+1)^2}$ $about \ \ z = -i$

15. $f(z) = \frac{1}{(z+1)(z+2)^2}$ $about\ z = 1$

Laurent Series :

Annulus : Given, $0 \leq R_1 < R_2$, we define the annulus centered at z_0 with radii R_1 *and* R_2 by

$$A = A(z_0, R_1, R_2) = \{ z : R_1 < |z - z_0| < R_2 \}$$



Laurent series generalizes Taylor series. Taylor series has positive integer powers and converges in a disk, whereas Laurent series is a series of positive and negative integer powers of $z - z_0$ and converges in an annulus.

Defination : Let a_n be complex numbers for $n = 0, \pm1, \pm2, \pm3, ...$ The doubly infinite series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ is called a Laurent series, and defined by

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \ , \quad \text{where } b_n = a_{-n}$$

Theorem 4 : Let $\{ f_n(z) \}$ be a sequence of functions continous on a domain D containing the contour C, and suppose that $\{ f_n(z) \}$ converges uniformly to $f(z)$. Then, the following holds :

$$\lim_{n\to\infty} \int_C f_n(z)\, dz = \int_C \left(\lim_{n\to\infty} f_n(z) \right) dz = \int_C f(z)\, dz$$

Proof: Let L be the length of C. Choose N large enough so that

$\left| f(z) - f_n(z) \right| < \frac{\epsilon}{L}$ for any $n \geq N$ and for all z on C.

Then, by ML inequality

$$\left|\int_C f(z)\,dz - \int_C f_n(z)\,dz\right| = \left|\int_C [\,f(z) - f_n(z)\,]dz\right| < \frac{\epsilon}{L} \times L = \epsilon$$

So, $\lim_{n\to\infty} \int_C f_n(z)\,dz = \int_C (\lim_{n\to\infty} f_n(z))dz$ *(Proved)*

Theorem 5 : If $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly to $S(z)$ on a domain D, then for any contour C in D the following holds :

$$\int_C \sum_{n=0}^{\infty} f_n(z)\,dz = \sum_{n=0}^{\infty} \int_C f_n(z)\,dz$$

Proof : Let $S_n(z) = \sum_{i=0}^{n} f_i(z)$.

Then, $\{ S_n(z) \}$ converges uniformly to $S(z)$.

$$\therefore \lim_{n\to\infty} \int_C S_n(z)\,dz = \int_C \left(\lim_{n\to\infty} S_n(z)\right) dz$$

$$or, \lim_{n\to\infty} \int_C \sum_{i=0}^{n} f_i(z)\,dz = \int_C S(z)\,dz$$

or, $\lim_{n\to\infty} \sum_{i=0}^{n} \int_C f_i(z)\,dz = \int_C \sum_{i=0}^{\infty} f_i(z)\,dz$

or, $\sum_{i=0}^{\infty} \int_C f_i(z)\,dz = \int_C \sum_{i=0}^{\infty} f_i(z)\,dz$ *(Proved)*

Laurent's Theorem : Let $f(z)$ be analytic in an annulus domain $A = \{ z : R_1 < |z - z_0| < R_2 \}$. Then, $f(z)$ can be represented by the Laurent series

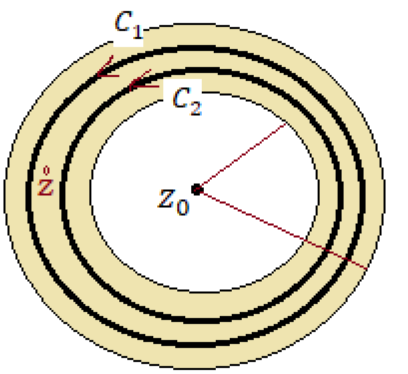
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad , z \in A$$

where , $a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$, $n = 0, 1, 2, \dots$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{-n+1}} d\xi \ , n = 1, 2, 3, \dots$$

and, C is any simple closed positively oriented contour around z_0 lying in A.

Proof:



Let z be any point in A.

We draw two positively oriented circles C_1, C_2 with radii r_1 and r_2 such that

$$R_1 < r_1 < |z - z_0| < r_2 < R_2 \ .$$

Then, the domain bounded by the circles C_1, C_2 lies in A and encloses the point z.

By Cauchy integral formula for doubly connected domains, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int_{C_2} \frac{f(\xi)}{\xi - z} d\xi \qquad \dots. (1)$$

For $\xi \in C_1$, we have $\left|\frac{z-z_0}{\xi-z_0}\right| < 1$ and

$$\frac{1}{\xi-z} = \frac{1}{(\xi-z_0)-(z-z_0)} = \frac{1}{\xi-z_0}\left[\frac{1}{1-\frac{z-z_0}{\xi-z_0}}\right]$$

$= \frac{1}{\xi-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^n$, which is uniformly convergent on C_1 .

So, $\frac{1}{2\pi i}\int_{C_1}\frac{f(\xi)}{\xi-z}d\xi = \frac{1}{2\pi i}\int_{C_1} f(\xi)\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\xi-z_0)^{n+1}}d\xi$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{C_1}\frac{f(\xi)}{(\xi-z_0)^{n+1}}d\xi\right](z-z_0)^n$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{C}\frac{f(\xi)}{(\xi-z_0)^{n+1}}d\xi\right](z-z_0)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad .... (2)$$

Now, for $\xi \in C_2$, we have $\left|\frac{\xi-z_0}{z-z_0}\right| < 1$ and

$$\frac{1}{z-\xi} = \frac{1}{(z-z_0)-(\xi-z_0)} = \frac{1}{z-z_0}\left[\frac{1}{1-\frac{\xi-z_0}{z-z_0}}\right]$$

$= \frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{\xi-z_0}{z-z_0}\right)^n$, which is uniformly convergent on C_2 .

So, $-\frac{1}{2\pi i}\int_{C_2}\frac{f(\xi)}{\xi-z}d\xi = \frac{1}{2\pi i}\int_{C_2} f(\xi)\sum_{n=0}^{\infty}\frac{(\xi-z_0)^n}{(z-z_0)^{n+1}}d\xi$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{C_2} f(\xi)(\xi-z_0)^n d\xi\right]\frac{1}{(z-z_0)^{n+1}}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_{C} f(\xi)(\xi-z_0)^{n-1} d\xi\right]\frac{1}{(z-z_0)^n}$$

$$= \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \qquad \dots (3)$$

From (1), (2) and (3), we get

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \ , \quad \text{where}$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \ , \quad n = 0, 1, 2, \dots$$

and, $$b_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{-n+1}} d\xi \ , \quad n = 1, 2, 3, \dots$$ *(Proved)*

Special Cases :

(1) If $f(z)$ is analytic everywhere in $B(z_0, R_2)$ expcept at z_0, then the Laurent series is valid in $0 < |z - z_0| < R_2$. Take $R_1 = 0$.

(2) If $f(z)$ is analytic in $B(z_0, R_2)$, then $\frac{f(\xi)}{(\xi - z_0)^{-n+1}}$ is analytic in $B(z_0, R_2)$, so that by Cauchy Goursat theorem $b_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{-n+1}} d\xi = 0$.

Hence, the Laurent series for $f(z)$ reduces to Taylor series for $f(z)$.

(3) We can write $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \ , \ z \in A$

where, $a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$ for all $n \in \mathbb{Z}$.

(4) Laurent series of an analytic function in an annular region can be differentiated term by term.

Since, Log z is not analytic in any annulus around 0, it cannot be represented by a Laurent series around 0.

Q1. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for

(i) $|z| < 1$ (ii) $1 < |z| < 3$ (iii) $|z| > 3$ (iv) $0 < |z+1| < 2$

Q2. Write all possible Laurent series for the function $f(z) = \frac{1}{z(z+2)^3}$ about $z = -2$.

Q3. Find the Taylor's and Laurent series which represents the function

$$\frac{z^2 - 1}{(z+2)(z+3)}$$

(i) when $|z| < 2$ (ii) when $2 < |z| < 3$ (iii) when $|z| > 3$

Q4. For the function $f(z) = \frac{2z^3+1}{z^2+z}$, find

(i) a Taylor series valid in the neighbourhood of the point *i.*

(ii) a Laurent series valid within the annulus of which centre is origin.

Q5. Find the Laurent series of $(z-3)\sin\frac{1}{z+2}$ around $z = -2$.

Q6. Find the Laurent series of $\frac{\sin z}{z^2}$ where $|z| > 0$.

Q7. Find the Laurent's expansion of $\frac{7z-2}{z(z-2)(z+1)}$ in the domains

(i) $|z| < 2$ (ii) $|z| > 3$ (iii) $2 < |z| < 3$