P-1

Scientific Notation, Scientific digits, The number of correct digits.

Any positive number a, can be represented as a terminating or nonterminating decimal:

$$a = \alpha_m 10^m + \alpha_{m-1} 10^{m-1} + \alpha_{m-2} 10^{m-2} + \cdots + \alpha_{m-n+1} 10^{m-n+1} + \cdots \quad (1)$$

where α_i are the digits of the number a ($\alpha_i = 0, 1, 2, \cdots 9$), the leading digit $\alpha_m \neq 0$ and m is an integer (the highest power of ten in the number a). For example,

$$3141.59\ldots = 3\times10^3 + 1\times10^2 + 4\times10^1 + 1\times10^0 + 5\times10^{-1} + 9\times10^{-2} + \cdots$$

Each unit occupies a specific position in the number a written as the decimal fraction (1) and has a definite value. The unit standing in the first position is equal to 10^m, that in the second position, 10^{m-1}, in the nth position, 10^{m-n+1}, etc.

Actual cases usually involve approximate numbers in the form of terminating decimals:

$$b = \beta_m 10^m + \beta_{m-1} 10^{m-1} + \cdots + \beta_{m-n+1} 10^{m-n+1} \quad (\beta_m \neq 0) \quad (2)$$

All retained decimal digits β_i ($i = m, m-1, \cdots, m-n+1$) are called <u>significant digits</u> of the approximate number b; note that some of them may be equal to zero (with the exception of β_m). In the decimal position system of representing the number b, one often has to introduce zeros at the beginning or at the end of the number. To illustrate

$$b = 7\times10^{-3} + 0\times10^{-4} + 1\times10^{-5} + 0\times10^{-6} = \underline{0.00}7010$$

or

$$b = 2\times10^{9} + 0\times10^{8} + 0\times10^{7} + 3\times10^{6} + 0\times10^{5} = 2,003,0\underline{00,000}$$

The underlined zeros are not significant.

Defn A significant digit of an approximate number is any non-zero digit, in its decimal representation, or any zero lying between significant digits or used as a place holder, to indicate a retained place. All other zeros of the approximate number that serve only to fix the position of the decimal point are not to be considered significant digits.

For example, in the number 0.002080 the first three zeros are not significant digits ~~be~~ since they serve only to fix the position of the decimal point and indicate the place values of the other digits. The other two zeros are significant digits since the first lies between the digits 2 and 8 and the second shows that we retain the decimal place 10^{-6} in the approximate number. If the last digit of 0.002080 is not significant, then the number must be written as 0.00208. From this point of view, the numbers 0.002080 and 0.00208 are not the same, because the former has four significant digits and the latter only three.

Let us introduce the notion of correct digits of an approximate number.

Defn We say that the first n significant digits of an approximate number are correct if the absolute error of the number does not exceed one half unit in the nth place, counting from left to right.

Thus, if for an approximate number 'a' represented by (1), which takes the place of an exact number A, it is known that

$$\Delta = |A-a| \leq \frac{1}{2} \times 10^{m-n+1}$$

then by definition the first n digits $\alpha_m, \alpha_{m-1}, \ldots, \alpha_{m-n+1}$ of this number is correct.

Now if an approximate number a having n correct digits is rounded off to n significant digits, the resulting new approximate number a_1 will, generally speaking, have n correct digits in the broad sense, Indeed, by ~~virtute~~ virtue of the inequality

$$|A-a_1| \leq |A-a| + |a-a_1|$$

the limiting absolute error of the number a_1 is made up of the absolute error of 'a' and the rounding error.

Ex-1. The number $a = 23.071937$ contains five valid digits. Find its absolute error.

Ans. Here $m=1$, $n=5$, and so we can take $\Delta_a = \frac{1}{2} \times 10^{1-5+1}$ $= 0.0005$ as the absolute error.

Ex-2 The absolute error of the number $a = 705.1978$ is $\Delta a = 0.3$. find out which of the digits of the number a are valid and round off the number a leaving only valid digits.

Ans. Here $m = 2$, $\Delta a = 0.3$ and n must be found from the inequality $\Delta a \le \frac{1}{2} \times 10^{m-n+1}$.

$$\therefore \quad 0.3 \le 0.5 \times 10^{3-n}.$$

A direct verification shows that the greatest n which satisfies this inequality is equal to 3 and the digit 5 is valid : $0.3 < 0.5 \times 10^{2-3+1}$ and the digit 1 is doubtful: $0.3 > 0.5 \times 10^{2-4+1}$

Consequently, the number $a = 705.1978$ has three valid digits. We round it off to three digits : $a_1 = 705$. Then the total error is equal to the sum of the initial error and the rounding error : $\Delta a = 0.3 + 0.1978 = 0.4978$, and so we can write $A = 705 \pm 0.4978$.

Theorem If a positive approximate number a has n correct digits in the narrow sence, the relative error δ of this number does not exceed $\left(\frac{1}{10}\right)^{n-1}$ divided by the first significant digit of the given number, or

$$\delta \le \frac{1}{\alpha_m \times 10^{n-1}}$$

where α_m is the first significant digit of the number 'a'.

Proof Let the number

$$a = \alpha_m 10^m + \alpha_{m-1} 10^{m-1} + \cdots + \alpha_{m-n+1} 10^{m-n+1} + \cdots \quad (\alpha_m \geq 1)$$

be an approximate number of an exact number A and let it be correct to n digits. By definition we then have

$$\Delta = | A - a | \leq \frac{1}{2} \times 10^{m-n+1}$$

whence $$A \geq a - \frac{1}{2} \times 10^{m-n+1}$$

This inequality is further strengthened if the number a is replaced by a smaller number $\alpha_m 10^m$:

$$A \geq \alpha_m 10^m - \frac{1}{2} \times 10^{m-n+1} = \frac{1}{2} \times 10^m \left(2\alpha_m - \frac{1}{10^{n-1}} \right) \quad \text{--} (1)$$

The right side of inequality (1) is a minimum for n=1.

Therefore

$$A \geq \frac{1}{2} \times 10^m (2\alpha_m - 1) \quad \text{--} (2)$$

or, since $2\alpha_m - 1 = \alpha_m + (\alpha_m - 1) \geq \alpha_m$

it follows that

$$A \geq \frac{1}{2} \alpha_m 10^m$$

Hence

$$\delta = \frac{\Delta}{A} \leq \frac{\frac{1}{2} \cdot 10^{m-n+1}}{\frac{1}{2} \alpha_m 10^m} = \frac{1}{\alpha_m \times 10^{n-1}}$$

Thus $$\delta \leq \frac{1}{\alpha_m \times 10^{n-1}} \quad \text{--} (3)$$

and the theorem is proved.