The error of the Interpolating Polynomial

Let $f(x)$ be a real-valued function on the interval $I=[a,b]$, and let $x_0, x_1, \dots, x_n$ be $(n+1)$ distinct points in I. With $p_n(x)$ the polynomial of degree $\le n$ which interpolates $f(x)$ at $x_0, x_1, \dots, x_n$, the interpolation error $e_n(x)$ of $p_n(x)$ is given by

$$e_n(x) = f(x) - p_n(x) \quad --- (1)$$

Let now $\bar{x}$ be any point different from $x_0, x_1, \dots, x_n$. If $p_{n+1}(x)$ is the polynomial of degree $\le n+1$ which interpolates $f(x)$ at $x_0, x_1, \dots, x_n$ and at $\bar{x}$, then $p_{n+1}(\bar{x}) = f(\bar{x})$, while by Newton's divided difference interpolating polynomial, we get

$$p_{n+1}(x) = p_n(x) + f[x_0; x_1; \dots; x_n; \bar{x}] \prod_{j=0}^{n} (x - x_j)$$

It follows that

$$f(\bar{x}) = p_{n+1}(\bar{x}) = p_n(\bar{x}) + f[x_0; \dots; x_n; \bar{x}] \prod_{j=0}^{n} (\bar{x} - x_j)$$

Therefore,

$$\text{for all } \bar{x} \ne x_0, x_1, \dots, x_n : \; e_n(\bar{x}) = f[x_0; \dots; x_n; \bar{x}] \prod_{j=0}^{n} (\bar{x} - x_j) \quad -(2).$$

Showing ~~that~~ the error to be "like the next term" in the Newton form.

We can't evaluate the right side of (2) without knowing the number $f(\bar{x})$. But as we now prove, the number $f[x_0; x_1; \dots; x_n; \bar{x}]$ is closely related to the $(n+1)$st derivative of $f(x)$, and using this information, we can at times estimate $e_n(\bar{x})$.

Th 1. Let $f(x)$ be a real-valued function, defined on $[a,b]$ and k times differentiable in (a,b). If $x_0, x_1, \dots, x_k$ are $(k+1)$-distinct points in $[a,b]$, then there exists $\xi \in (a,b)$ such that

$$f[x_0; x_1; \dots; x_k] = \frac{f^{k}(\xi)}{\lfloor k} \quad -- (3).$$

of For $k=1$, this is just the MVT for derivatives. For the general case, observe that the error function $e_k(x) = f(x) - p_k(x)$ has (at least) the $k+1$ distinct zeros $x_0, x_1, --, x_k$ in $I=[a,b]$. Hence, if $f(x)$, and therefore $e_k(x)$ is k-times differentiable on (a,b), then it follows from Rolle's theorem that $e'(x)$ has at least k-zeros in (a,b); hence $e''(x)$ has at least $(k-1)$ zeros in (a,b), and continuing in this manner, we finally get that $e_k^k(x)$ has at least one zero in (a,b). Let ξ be one such ~~error~~ zero. Then

$$0 = e_k^k(\xi) = f^k(\xi) - p_k^k(\xi).$$

On the other hand, we know that, for any x

$$p_k^k(x) = f[x_0; x_1; --; x_k]\cdot \underline{|k}$$

since, by defn., $f[x_0; x_1; --; x_k]$ is the leading co-efficient of $p_k(x)$.

$$\therefore \quad p_k^k(\xi) = f^k(\xi) \quad \text{gives}$$

$$f[x_0; x_1; --; x_k]\cdot \underline{|k} = f^k(\xi)$$

$$\text{or,} \quad f[x_0; x_1; ---; x_k] = \frac{f^k(\xi)}{\underline{|k}}, \quad \xi \in (a,b)$$

— x —

Now using (2) & (3) we get

$$e_n(\bar{x}) = \frac{f^{(k+1)}(\xi)}{\underline{|k+1}} \prod_{j=0}^{n} (\bar{x} - x_j) \quad \text{—— (4)}$$

Since $e_n(x_0) = e_n(x_1) = ---- = e_n(x_n) = 0$, dropping the bar in (4), we get in general

$$e_n(x) = \frac{f^{(k+1)}(\xi)}{\underline{|k+1}} \prod_{j=0}^{n} (x - x_j) \quad \text{—— (5)}$$