Uniform continuity

The notion of continuity of a function is essentially local in character in that reference to continuity of a function at a point is meaningful, the same is not the case in respect of uniform continuity. The notion of uniform continuity of a function is global in character as we can talk of uniform continuity of a function only over its domain.

Let f be a function with domain D. Let $a \in D$. Now f is continuous at a if for every $\epsilon > 0$ $\exists$ some $\delta > 0$ s.t.

$$|f(x) - f(a)| < \epsilon \quad \forall x \in D \text{ s.t. } |x - a| < \delta.$$

Suppose that f is continuous at each point of D and $\epsilon > 0$ is any given number. Then there exists a $\delta > 0$ corresponding to each point of the domain D. Now the question arises "Does there exist δ which holds uniformly for every point of the domain?". It is not possible for every case. So in general δ depends on both ϵ

as well as the point on which we consider continuity.

Defn:- A function f defined on an interval I is said to be uniformly continuous on I, if for any $\epsilon > 0$ $\exists$ $\delta > 0$ s.t. $|f(x) - f(y)| < \epsilon$ for any $x, y \in I$ where $|x - y| < \delta$.

Note - A function is not uniformly continuous on I, if $\exists$ some $\epsilon > 0$ for which no $\delta > 0$ serves, i.e, for any $\delta > 0$, there exist $x, y \in I$ s.t. $|f(x) - f(y)| \geq \epsilon$ when $|x - y| < \delta$.

Theorem 1 - Every uniformly continuous function on an interval is continuous there, but the converse is not true.

Proof - Suppose f is uniformly continuous on I. Then given $\epsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x - y| < \delta \quad \text{---(1)}$$

Let, $a \in I$. Taking $y = a$ in (1) we have $|f(x) - f(a)| < \epsilon$ when $|x - a| < \delta$.
$\Rightarrow$ f is continuous at a.

since a is arbitrary pt. of I, so f is continuous on I.

<u>Theorem 2</u> – If a function is continuous in a closed interval then it is uniformly continuous there.

Proof – If possible let f is continuous in $[a, b]$ but not uniformly continuous there.

So there exists some ϵ_0 such that for every $\delta > 0$ there exists a pair of points x, y in $[a, b]$ such that $|x - y| < \delta$, but $|f(x) - f(y)| \geq \epsilon_0$.

Taking $\delta = 1, 1/2, 1/3, \ldots$ successively.

$\exists\, x_1, x_1'$ in $[a, b]$ s.t. ~~$|x_1 - x_1'| < \delta$, $|f(x_1) - f(x_1'$~~

$|x_1 - x_1'| < 1$, but $|f(x_1) - f(x_1')| \geq \epsilon_0$

$|x_2 - x_2'| < 1/2$ but $|f(x_2) - f(x_2')| \geq \epsilon_0$

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$|x_n - x_n'| < 1/n$, but $|f(x_n) - f(x_n')| \geq \epsilon_0$

Hence we have two sequences $\{x_n\}$ and $\{x_n'\}$ in $[a, b]$

for which $|x_n - x_n'| < \frac{1}{n}$, but $|f(x_n) - f(x_n')| \geq \epsilon_0$

Now $\{x_n\}$ is bounded (as $a \leq x_n \leq b$) for every n and so it has a limit point α. Therefore there exists a subsequence $\{x_{n_k}\}$ converging to α.

Thus $\alpha = \lim\limits_{k \to \infty} x_{n_k} \geq a$ as $x_{n_k} \geq a \;\forall\, n_k$

and $\alpha = \lim\limits_{k \to \infty} x_{n_k} \leq b$ as $x_{n_k} \leq b \;\forall\, n_k$.

$\therefore a \leq \alpha \leq b$ and so f is continuous at α.

Also
$$|\alpha - x'_{n_k}| = |\alpha - x_{n_k} + x_{n_k} - x'_{n_k}|$$
$$\leq |\alpha - x_{n_k}| + |x_{n_k} - x'_{n_k}|$$
$$< |\alpha - x_{n_k}| + \frac{1}{n_k}$$

As $k \to \infty$, $n_k \to \infty$ and $x_{n_k} \to \alpha$

$\therefore |\alpha - x'_{n_k}| \to 0 \Rightarrow x'_{n_k} \to \alpha$ as $k \to \infty$.

Now, $|f(x_{n_k}) - f(x'_{n_k})| \to |f(\alpha) - f(\alpha)| = 0$

[$\because f$ is continuous at α]

which contradicts the fact that $|f(x_{n_k}) - f(x'_{n_k})| \geq \epsilon_0$.

So, f is uniformly continuous in $[a, b]$

Th 3 Let $f: I \to R$ be uniformly continuous on I, then for every cauchy sequence $\{x_n\}$ in I, $\{f(x_n)\}$ is cauchy sequence in R.

Soln:- By definition given any $\epsilon > 0$ $\exists\ \delta > 0$ s.t. for any $x, x' \in I$, $|x - x'| < \epsilon \Rightarrow |f(x) - f(x')| < \epsilon$.

Given $\{x_n\}$ is cauchy sequence in I. $\therefore \exists$ a natural no N s.t. $|x_m - x_n| < \delta$ when $m, n > N$.

$\therefore \forall\ m, n > N \Rightarrow |x_m - x_n| < \delta \Rightarrow |f(x_m) - f(x_n)| < \epsilon$ in R. $\therefore \{f(x_n)\}$ is a cauchy sequence in

Note This theorem is very important to prove non-uniform continuity of a function in an interval I.

Ex Show that $f(x) = \frac{1}{x}$ is continuous in $(0,1)$ but not uniformly continuous there.

Soln:- x is continuous in $(0,1)$ and the interval $(0,1)$ does not contain $x = 0$. $\therefore \frac{1}{x}$ is continuous in $(0,1)$.

Now for any $\delta > 0$ we can choose an natural no. m s.t. $\frac{1}{m} < \delta$ $\forall\ n \geq m$. Let, $x = \frac{1}{m}$, $y = \frac{1}{2m}$

$\therefore x, y \in (0,1)$. $|x - y| = \frac{1}{2m} < \delta$

But $|f(x) - f(y)| = |m - 2m| = m$ can not be made smaller than $\epsilon > 0$ choosen arbitrarily. $\therefore f(x) = \frac{1}{x}$ is not uniformly continuous in $(0,1)$

Q No

Alnative (2) If Possible let f be uniformly continuous in (0,1)

Then for every cauchy sequence $\{x_n\}$ in (0,1) the sequence $\{f(x_n)\}$ will be a cauchy sequence.

Take $x_n = \frac{1}{n}$ $\{\frac{1}{n}\}$ is cauchy sequence, but $\{f(x_n)\} = \{n\}$ which is not cauchy

[illegible] $(x > 0)$

[illegible]

[illegible]

[illegible]

[illegible]

$\therefore$ [illegible] is not uniformly continuous on R^+.

[illegible] $\{x_n\}$ where $x_n = \frac{2}{n\pi}$, $n \in N$ which is cauchy sequence.

but $\{f(x_n)\} = \{1, 0, -1, 0, \ldots\}$ which is not a cauchy sequence

So $f(x)$ is not uniformly convergent.